COMPARING CATEGORIES OF DOMAINS

by

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Abstract. We discuss some of the reasons for the proliferation of categories of domains suggested for the mathematical foundations of the Scott-Strachey theory of programming semantics. Five general conditions are presented which such a category should satisfy and they are used to motivate a number of examples. An attempt is made to survey some of the methods whereby these examples may be compared and their relationships expressed. We also ask a few mathematical questions about the examples.

1. Introduction.

A great variety of mathematical structures have been proposed for use as semantic domains for programming languages. We focus on one line of investigation which uses certain classes of partially ordered sets and aims to give a semantics which is denotational in nature. This approach was introduced by Dana Scott and Chris Strachey in the late sixties ([24], [30]) and it remains an area of active research today. The original category used by Scott and Strachey had complete lattices as objects and monotone functions that preserve least upper bounds of directed collections as arrows. But in the decade and a half since their work a host of other closely related categories have been investigated. Discussing the reasons that these alternatives have been suggested and the relationships between the different categories is the goal of the current document. A secondary objective is to ask a few mathematical questions about the categories. Most of the questions mentioned are not motivated by any particular problem in programming semantics. It is hoped, however, that they will evoke the curiosity of the reader as they have that of the author.

The paper is divided into four sections and an appendix. Section two discusses some of the conditions from programming semantics which motivate the choice of a category of domains. A collection of five such conditions are enumerated and we discuss how these conditions are satisfied to one degree or another by specific categories. Section three discusses what might be called "distinguishing conditions" on categories. The most important of these is Smyth's Theorem and

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we state some of its generalizations. The fourth section introduces the categories of "continuous domains" which are a current area of investigation. Proofs of most of the theorems stated below can be found scattered throughout the literature (see, in particular, [4], [17] and [29]). A few short proofs that do not require much background have been included in an appendix. A result whose proof may be found there is marked with an asterisk (*).

The reader is assumed to be familiar with the following concepts from category theory: category, functor, object, arrow, product and coproduct, terminal and initial objects, equivalence and isomorphism between categories, inverse limit, and continuous functor. Definitions may be found in any of the standard references on category theory ([1], [5], [13]).

2. In search of the perfect category of domains.

Basic definitions and notation. A *poset* is a set with a binary relation $\sqsubseteq$ which is reflexive, anti-symmetric and transitive. If D is a poset, a subset $M \subseteq D$ is *directed* if every finite subset of M has a bound in M. A poset D is *directed complete* if every directed $M \subseteq D$ has a least upper bound $\bigsqcup M$ in D. A function $f : D \to E$ is *Scott continuous* if it is monotone and $f(\bigsqcup M) = \bigsqcup f(M)$ for each directed $M \subseteq D$. The term "continuous" comes form the fact that it is possible to define a topology on a dcpo which makes these directed lub preserving maps exactly the continuous functions. If D is a dcpo then a subset $\mathcal{O} \subseteq D$ is said to be *Scott open* if

1. If $x \in \mathcal{O}$ and $x \sqsubseteq y$ then $y \in \mathcal{O}$.

2. If M is directed and $\bigsqcup M \in \mathcal{O}$ then $M \cap \mathcal{O} \neq \emptyset$.

These open sets form a T_0 topology on D called the *Scott topology.*

We denote by **DCPO** the category of directed complete posets and continuous functions. All of the domains that we consider below will be dcpo's. We therefore adopt the following convention: unless otherwise stated, every category **C** is assumed to be a full subcategory of **DCPO**. Here are a few examples:

- Let S be any set. Order S discretely, *i.e.* $x \sqsubseteq y$ iff $x = y$. These discretely ordered posets are all dcpo's. If S and T are discretely ordered posets then *any* function $f : S \to T$ is continuous.

- Any finite poset is a dcpo.

- If α is an ordinal then $\alpha + 1$ is a dcpo.

- If S is a set then the powerset of S, ordered by set inclusion, is a dcpo.

- The extended reals (*i.e.* the reals under the usual ordering with a largest and smallest element added) form a dcpo.

DCPO is a rather "large" category. Indeed, the discretely ordered posets mentioned above form a full subcategory $\mathbf{Set} \subseteq \mathbf{DCPO}$ which is isomorphic to the category of sets. There are, however, familiar partially ordered sets which are *not* dcpo's. For example, the rational numbers do not form a dcpo for two reasons. First, there would need to be a top (greatest) element to act as a least upper bound for the whole set of rationals (since any linear order is directed). But the second and less trivial reason is that the irrationals are "missing". In short, the rationals are simply not dense enough to be a dcpo.

The class of dcpo's having a bottom (least) element is of considerable importance in domain theory. If a domain D has a least element then it is usually denoted by the symbol $\perp_D$ (bottom) and the subscript is dropped when there is no likelihood of confusion. A domain with a least element is said to be *pointed.* If $\mathbf{C} \subseteq \mathbf{DCPO}$ then $\mathbf{C}_\perp$ is defined to be the full subcategory of **DCPO** which has as objects those dcpo's in **C** which have a least element.

Operations on dcpo's. In giving the semantics of a programming language it is necessary to utilize a number of operators on dcpo's to build up the desired data types from given primitive data types (such as integers or booleans). Here are some of the operators commonly used:

- Product: $D \times E$,
- Function space: $[D \to E]$,
- Disjoint sum: $D + E$,
- Separated sum: $D \oplus E$,
- Lift: $D_\perp$.

The product and function space are ordered coordinatewise and *only* continuous functions are included in $[D \to E]$. To get the disjoint sum of domains D and E, one "colors" D and E so their elements cannot be confused and then takes the union of the two (colored) posets. This is different from the separated sum which is used in most of the literature on denotational semantics. It is defined by $D \oplus E = (D+E)_\perp$ where $F_\perp$ is the result of adjoining a new bottom element to a domain F. All of the operators listed above except $+$ are closed on $\mathbf{DCPO}_\perp$. The first three operators ($\times$, $\to$, and $+$) are also closed on **Set**. There is another commonly used flavor of sum which is called the *coalesced sum.* If $D_\perp$ and $E_\perp$ are pointed dcpo's, then their coalesced sum is $D \oplus E$. Note that the coalesced sum only makes sense on pointed domains. Between the disjoint, separated and coalesced sums, the disjoint sum is certainly the most elegant and natural mathematically. In **DCPO**, it is the *categorical coproduct* and consequently has several nice relationships with the operators $\times$ and $\to$. For example, the following isomophism holds for all domains D, E, F: $D \times (E + F) \cong (D \times E) + (D \times F)$. Moreover, $+$ and $\to$ are related by the following isomorphism: $[(D + E) \to F] \cong [D \to F] \times [E \to F]$.

Cartesian closure. There is an important categorical condition which has arisen as being particularly significant for domain theory. This is the notion of a *cartesian closed category.* In mathematics, cartesian closed categories are somewhat rare. No doubt the best known example is the category of sets and functions. For sets there is an isomorphism

$$\text{curry} : [(D \times E) \to F] \cong [D \to [E \to F]]$$

defined as follows. If $f : D \times E \to F$ and $(x, y) \in D \times E$, then $\text{curry}(f)(x)(y) = f(x, y)$. It is well outside the scope of the current document to discuss all of the reasons that ccc's are important for programming semantics. Besides, there is a wealth of literature available on the subject ([9], [10], [20]). Most of it is concerned with the connection between ccc's and models of typed and untyped λ-calculus. For our purposes, a (full) subcategory $\mathbf{C} \subseteq \mathbf{DCPO}$ is cartesian closed iff it is closed under the product $(\times)$ and function space $(\to)$ operations. The objects of $\mathbf{C}$ should also include the one point domain 1. This is a specialization of the actual definition which one finds in category theory books. We now offer up the first of our conditions on a category of domains:

> **CONDITION 1:** The category must be closed under the desired operators. (Particularly product and function space operators: cartesian closure is a good technical condition.)

And we have the following:

Theorem 1 **DCPO** *is a cartesian closed category.* **Set** *and* $\mathbf{DCPO}_\perp$ *are also cartesian closed.*

In fact, **DCPO** and **Set** are endowed with the additional nicity of being closed under the disjoint sum operation $+$. Moreover, the empty poset 0 is an identity for $+$ which is initial in the **DCPO**. A cartesian closed category which has a coproduct and initial object is said to be *bicartesian closed.*

Equational specification. Much of the essence of the denotational semantics of programming languages involves the equational specification of meanings for language constructs. For example, one might write something like to following equation to specify the meaning of a while loop:

$$[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]\sigma = \begin{cases} \sigma & \text{if } [\![B]\!]\sigma = \text{false;} \\ [\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]([\![C]\!]\sigma) & \text{if } [\![B]\!]\sigma = \text{true.} \end{cases}$$

were σ is a machine state. One needs a proper theory of what sort of domains are used in interpreting the various operators in the above equation in order to get good general conditions for when such an equation has a *canonical* solution. When the data types involved are dcpo's and the functions are continuous, the following theorem is the key fact:

Theorem 2 *Suppose D is a dcpo and $f : D \to D$ is continuous. If $x \sqsubseteq f(x)$ for some $x \in D$ then there is a least $y \in D$ such that $x \sqsubseteq y$ and $y = f(y)$. In particular, if D has a $\bot$ then f has a least fixed point.*

The proof of the theorem is not difficult. If $x \sqsubseteq f(x)$ then the directed completeness of D allows us to find a point $y = \bigsqcup f^i(x)$. But f is continuous, so $f(y) = f(\bigsqcup f^i(x)) = \bigsqcup f^{i+1}(x) = y$.

It is, moreover, possible to specify *data types* using equations (or, to be a bit more precise, isomorphisms) Here are a few examples:

- Simple binary tree: $T \cong T \oplus T$.
- S-expressions: $T \cong At \oplus (T \times T)$.
- λ-calculus: $D \cong At \oplus [D \to D]$.
- Finitely branching trees T and forests F of finitely branching trees:

$$T \cong At \times F$$
$$F \cong 1 \oplus (T \times F).$$

A great deal of thought has been given to the issue of specifying data types in this way. Papers such as [11], [29] and [31] seek to find a general categorical framework within which one may find canonical solutions to such equations. Typically this involves considering categories with some order structure on their hom sets and finding a fixed point of a continuous functor F by locating an initial object in a category defined from F and $\mathbf{C}$. We now sketch the way this treatment applies to $\mathbf{DCPO}$ (which is, in any event, a primary motiviating example in these categorical treatments.) The following notion is central to the success of the approach:

Definition: Let D and E be dcpo's. A continuous function $f : E \to D$ is a *retraction* if there is a continuous $g : D \to E$ such that $f \circ g = \mathrm{id}_D$. If g satisfies the further condition that $g \circ f \sqsubseteq \mathrm{id}_E$ then f is called a *projection* and g an *embedding*.

Remark: if a dcpo D is pointed then the (unique) function $!_D : D \to 1$ is a projection. If $\mathbf{C} \subseteq \mathbf{DCPO}$, let $\mathbf{C}^P$ be the category having the same objects as $\mathbf{C}$ and having projections as arrows. The remark amounts to saying that the domain 1 is terminal in the category $\mathbf{DCPO}_\bot^P$. The significance of this fact comes from the following fixed point existence theorem:

Theorem 3 *Suppose $\mathbf{C} \subseteq \mathbf{DCPO}$ and $\mathbf{C}^P$ has limits for countable inverse systems. If $F : \mathbf{C}^P \to \mathbf{C}^P$ is a continuous functor and there is a domain $I \in \mathbf{C}$ and a projection $p : F(I) \to I$, then there is a dcpo $D \in \mathbf{C}$ such that $D \cong F(D)$.*

Because of its importance, let us look at a brief sketch of the proof of this theorem. If $p : F(I) \to I$ then because F is a functor there is a projection $F(p) : F^2(I) \to F(I)$. Continuing in this way, one builds an inverse system:

$$I \xleftarrow{p} F(I) \xleftarrow{F(p)} F^2(I) \xleftarrow{F^2(p)} \cdots$$

Now, let $D = \varprojlim F^i(I)$ be the limit of this system. Then by continuity we compute: $F(D) = F(\varprojlim F^i(I)) \cong \varprojlim F^{i+1}(I) \cong D$. Note the similarity between this proof and the proof of Theorem 2.

Suppose that $\mathbf{C}_\perp$ is a category of pointed domains with $1 \in \mathbf{C}$ and suppose $\mathbf{C}_\perp^P$ has limits for countable inverse systems. If $F : \mathbf{C}_\perp^P \to \mathbf{C}_\perp^P$ is continuous, then it has a fixed point because the domain 1 can serve as the object I in the theorem. Of course, this situation is highly analogous to the one which arose when we set down conditions in Theorem 2 for the existence of a fixed point of a continuous *function.* The object I is like the point x such that $x \sqsubseteq f(x)$. A category of pointed dcpo's is therefore like a domain with a least element. This analogy between continuous functions on a domain and continuous functors on a category of domains is often helpful.

It is a well-known fact from the domain theory literature that all of the operators on dcpo's that we have mentioned so far can be made into continuous functors on $\mathbf{DCPO}^P$ by a proper choice of action on morphisms. The most interesting example is the function space operator $\to$. If $p_D : D \to D'$, and $p_E : E \to E'$ are projections then we define a projection

$$[p_D \to p_E] : [D \to E] \to [D' \to E']$$

by letting $[p_D \to p_E](f) = p_E \circ f \circ q_D$ where q_D is the embedding that corresponds to p_D. Proof that this defines a continuous functor can be found in [4] or [29].

The situation with fixed points can be a bit problematic, however. Theorem 3 can be used to show that a fixed point *exists* but it does not insure that the solution constucted will be non-trivial. For example, the operator $F(X) = [X \to X]$ is closed on pointed dcpo's so it has a fixed point. But the result that one gets from using $I = 1$ is just the one element domain itself! The solution to this problem is to start with a non-trivial domain I. If I is *pointed* then it is possible to find a projection $p : [I \to I] \to I$ so there are also plenty of solutions for $X \cong [X \to X]$. This also works for getting non-trivial solutions to other equations like $X \cong X \times X$. One can show that for any pointed dcpo I, there is a projection $p : I \times I \to I$ so non-trivial solutions to this equation are also abundant.

Actually, the functor $\times$ is continuous on dcpo's with *arbitrary* continuous functions as arrows. A fact similar to that expressed by Theorem 3 can be proved and used to find still more fixed points for the $X \times X$ operator using continuous functions $f : I \times I \to I$. This is not true for the function space operator, however. The functor $\to$ on $\mathbf{DCPO}$ is contravariant in its second argument and therefore is not continuous. In fact this is one of the primary reasons for using *projections* as arrows. As noted above, with projections as arrows, it is possible to get an action of $\to$ on arrows making it a continuous functor. Note, however, that no discrete set other than 1 can be projected onto its function space. Moreover, no finite poset (other than 1) has this property. In short, there can be a problem in deciding of a given equation whether Theorem 3 will help. For example, the equation $X \cong X + X$ has many solutions since any infinite discrete set X will work. However, the equation

$X \cong 1 + [X \to X]$ has *no* solution. (To prove this count the number of disjoint components in X and $1 + [X \to X]$.) The equation $X \cong [X \to 0]$ also has no solution. The following question is suggested: *Is there a procedure for effectively deciding whether an operator built up from constants and the operators $+, \times, \to$ has a fixed point in* **DCPO***?*

We summarize the need for solutions to recursive equations as our second condition:

> **CONDITION 2:** There should be (canonical) solutions for recursive equations.

The condition is to be taken as expressing a *generally desirable* property. It is not really necessary that *all* equations have fixed points but there should be good conditions for when an equation does have a solution. We offer the claim that **DCPO** (or at least $\mathbf{DCPO}_\perp$) meets the condition quite nicely.

Since the hypothesis in Theorem 3 requires the existence of *countable* inverse limits (*i.e.* limits for countable inverse systems), we adopt the convention that all inverse limits that we mention below are limits of countable systems. We might have adopted a similar convention for dcpo's, requiring only that *countable* directed collections have a least upper bound. This would not interfere with Theorem 2 because the proof uses only the existence of the least upper bound of an ω-chain. In much of the literature this weaker condition of *chain completeness* is used instead of arbitrary directed completeness. For the purposes of this paper, it makes no difference which notion is used.

Computability. One of the most natural conditions for a category of semantic domains for compter programming languages to satisfy is that there be a good notion of computability. In particular, it should be possible to say what it means for a domain to be *effectively given* and what it means for a function between effectively given domains to be *computable.* A definition of a *computable element* would, one expects, have the following property: a *function* $f : D \to E$ between effectively given domains D and E is computable iff the *element* $f \in [D \to E]$ is computable. One should also have a satisfactory notion of *computable data type constructor* or, more precisely, *computable functor.* One hopes that the constructors we have mentioned so far turn out to be computable and that important arrows associated with those operators turn out to be computable.

To do this, it appears that **DCPO** is *not* the right category. For a proper notion of computability, a domain needs to have some kind of basis of "finite" or "one-step computable" elements which can be used to approximate the *infinite* elements of the domain. Stated vaguely we ask that the following be satisfied:

> **CONDITION 3:** There should be a flexible and intuitive theory of computability through finitary approximation.

The details of how to derive this theory of computability are beyond the scope of this paper. It has been the object of intense study over the last decade. The topic is discussed for particular categories in [19], [21], [25], and [32]. The issue of getting effective presentations for domains that are specified as fixed points is the central topic of [7], [8], and [33]. McCarty [14] studies the use of intuitionistic set theory for getting a theory of computability. All of these approaches deal in some degree with subcategories of **DCPO** for which the objects have a basis. Several of the studies use the following notion:

Definition: Let D be a dcpo. An element $x \in D$ is said to be *finite* if whenever $M \subseteq D$ is directed and $x \sqsubseteq \bigsqcup M$, then $x \sqsubseteq y$ for some $y \in M$. Let D^0 denote the set of finite elements of D. Then D is said to be *algebraic* if for every $x \in D$, $M = D^0 \cap \downarrow x$ is directed and $x = \bigsqcup M$. If D^0 is countable, then D is said to be ω-algebraic.

The above-mentioned literature shows how to define a quite satisfactory notion of computability for ω-algebraic dcpo's and continuous functions between them. Such a treatment depends crucially upon the countability of the basis for the domain. We therefore adopt the following convention: all algebraic dcpo's will be assumed to have a countable basis. Accordingly, henceforth, **Set** is the category of *countable* discretely ordered posets. We define **Alg** to be the category of (ω-)algebraic dcpo's.

Computability and Condition 1. Unfortunately, **Alg** is *not* cartesian closed and therefore does not satisfy our closure condition. The problem is that there are algebraic dcpo's D such that $[D \to D]$ fails to be algebraic (see [27]). It is therefore necessary to look for cartesian closed subcategories of **Alg**. Fortunately, there are quite a few of these. We need a few terms:

Definition: A poset $A \neq \emptyset$ is *bounded complete* if every finite bounded $u \subseteq A$, has a least upper bound. A is *coherent* if every finite $u \subseteq A$ which is pair-wise bounded has a least upper bound.

We use the following notation:

- **AlgLat** = algebraic lattices,
- **CohAlg** = coherent algebraic dcpo's,
- **BCAlg** = bounded complete algebraic dcpo's.

The main fact is this:

Theorem 4 **AlgLat, CohAlg, BCAlg** *are cartesian closed and have the corresponding categories with projections as arrows have inverse limits.*

Simplicity. Certainly one would like to work with a category which is easy to describe and understand. Proving the basic properties (such as cartesian closure and existence of desirable arrows) should also be straight-forward. Part of the reason that the three categories in the theorem above are the ones most commonly used in programming semantics today is the degree to which they satisfy the following:

CONDITION 4: The category should be natural to motivate and simple to describe.

Of course, what one considers "natural" or "simple" is a matter of taste and domain theory has been criticized for failing to do more toward satisfying Condition 4. Scott ([19], [21], [22], [23]) has made an effort to correct this (perceived) problem with his original treatment [18]. This has been followed up by other researchers (for example, [2], [4] and [33]) and some progress has been made but the final word has probably not yet been written.

Basic Data Types. The reader may be curious about why *three* cartesian closed subcategories of **Alg** have been considered. The algebraic lattices were given a rather thorough treatment by Scott [19] but some domain theorists complained that the existence of a top element in the domains was unnatural and inconvenient. For example, the infinite discrete set N must have a bottom element and top element added to it to get a lattice. The resulting domain $N_\perp^\top$ is then used as the natural numbers data type. The intuition is that the bottom element is the value of a divergent computation. Plotkin [16] urged the use of the category of coherent algebraic dcpo's as an alternative and gave a treatment for this class which is analogous to Scott's treatment of the algebraic lattices. For example, the natural number data type can be taken as $N_\perp$ rather than $N_\perp^\top$ because the former is coherent (although not a lattice). Subsequently, Scott ([21], [22], [23]) has urged the use of the bounded complete algebraics on the grounds that the troublesome top element is avoided and **BCAlg** is larger and simpler than **CohAlg**. We summarize (part of) this issue in the following:

CONDITION 5: The category must possess the desired basic data types (or facsimiles thereof).

3. Profinite domains and Smyth's Theorem.

One problem with the categories **AlgLat, CohAlg, BCAlg** is that there are operators such as $+$ and the convex powerdomain ([15], [26] [28]) which are not closed on these classes. A noteworthy category which *is* closed under these operators is the category of *profinite* domains. These are defined as follows:

Definition: A dcpo is (ω-)*profinite* if it is isomorphic to a countable inverse limit (in $\mathbf{DCPO}^P$) of finite posets.

One drawback to this definition is that it does not define the profinites *intrinsically.* In other words, to tell whether a domain is profinite requires that one locate an inverse system of finite sets. It would be better to find a condition on a domain D which shows that D is profinite without reference to other posets. Such an intrinsic characterization is provided by the following:

Theorem 5 *Let D be a dcpo and let M be the set of continuous functions $p : D \to D$ such that $p = p \circ p \sqsubseteq \mathrm{id}_D$ and $im(p)$ is finite. Then D is profinite if M is countable, directed and $\bigsqcup M = \mathrm{id}_D$.*

Several other intrinsic characterizations of profinite domains are possible ([15], [4]). The point is that the profinites are not far from satisfying Condition 4. The category **P** of profinite domains is also quite large:

Theorem 6 **AlgLat** $\subseteq$ **CohAlg** $\subseteq$ **BCAlg** $\subseteq$ $\mathbf{P}_\perp \subseteq \mathbf{P} \subseteq$ **Alg** *and none of these inclusions is reversible.*

Indeed, there are many domains which are profinite but not bounded complete. For example, *all* finite posets are in **P**. However, no infinite discrete set is profinite. **P** also has limits for proving the existence of fixed points:

Theorem 7 $\mathbf{P}^P$ *and* $\mathbf{P}_\perp^P$ *have limits for inverse systems.*

It has shown [4] that it is possible to enumerate those operators built up from constants and $+, \times, \to$ which have fixed points. However, it is not known whether this fixed point existence property is decidable.

The primary reason for interest in the profinites is the following fact:

Theorem 8 *Suppose* $F : \mathbf{DCPO}^P \to \mathbf{DCPO}^P$ *is a continuous functor. If* $F(A)$ *is finite whenever* A *is finite then* $F(D)$ *is profinite whenever* D *is profinite.*

This says that **P** has rather robust closure properties. There is an obvious generalization of the theorem to multiary functors, so the theorem may be applied to show, for example, that **P** is bicartesian closed. The subcategory $\mathbf{P}_\perp$ of pointed profinite domains is also a pleasing one. On can prove closure results for it by using the following observation. Suppose $F : \mathbf{DCPO}^P \to \mathbf{DCPO}^P$ is a continuous functor. For finite posets A, if $F(A)$ has a $\perp$ whenever A does then $F(D)$ has a $\perp$ whenever D does. In particular, $\mathbf{P}_\perp$ is cartesian closed (but not bicartesian closed).

Other bicartesian closed categories. The following observation of Scott shows the existence of quite a few bicartesian closed categories of domains:

Theorem* 9 *Suppose* $\mathbf{C} \subseteq \mathbf{DCPO}_\perp$ *is cartesian closed. If* $\mathbf{C}'$ *is the category whose objects include* 0 *and all dcpo's of the form*

$$D_1 + \cdots + D_n$$

where $D_1, \ldots, D_n$ *are in* $\mathbf{C}$*, then* $\mathbf{C}'$ *is bicartesian closed. Moreover, if* $\mathbf{C} \subseteq \mathbf{Alg}_\perp$ *then* $\mathbf{C}' \subseteq \mathbf{P}$*.*

However, as was noted in the previous section, none of these categories has solutions for equations like $X \cong 1 + [X \to X]$. *Is there an interesting bicartesian closed category on which operations built up from* $\times$*,* $\to$*,* $+$ *etc. all have fixed points?* One stab at answering the question is to consider non-trivial complete Heyting algebras. These are bicartesian closed categories for which the fixed point theorem applies. Unfortunately, the negation operator, $\neg X = [X \to 0]$, is not continuous and does not have a fixed point in any such algebra. Perhaps the question is expressed a bit too strongly. There are, after all, quite a few strange looking operators that one can build using 0.

Smyth's Theorem. The intuition that the profinites form a "large" subcategory of **Alg** is confirmed by a theorem of Smyth [27]. Smyth proved the following conjecture of Plotkin:

Theorem 10 *(Smyth) If D and $[D \to D]$ are pointed and algebraic then D is profinite.*

This is especially significant in light of Condition 1 because it yields the following:

Corollary 11 *(Smyth) If $\mathbf{C} \subseteq \mathbf{Alg}_\perp$ is cartesian closed then $\mathbf{C} \subseteq \mathbf{P}_\perp$.*

Warning: Smyth's Theorem may *fail* if the domain D is not countably based. The theorem also leaves open several related questions. It is, for example, not difficult to show that one of the hypotheses may be weakened slightly:

Theorem* 12 *If D is a pointed dcpo and $[D \to D]$ is algebraic then D is profinite.*

With some care, the proof of Smyth's Theorem can be modified [4] to extend the result to algebraic dcpo's which are not pointed:

Theorem 13 *If D and $[D \to D]$ are algebraic then D is profinite.*

However, the author does not know an answer to the following question: *If D is a dcpo and $[D \to D]$ is algebraic then is D profinite?*

Strongly algebraic domains and partial functions. Another approach to finding a good category of domains involves not only restricting the *objects* but also working with a different kind of *arrow.* We make the following definition and observation:

Definition: A dcpo D is *strongly algebraic* if $D_\perp$ is profinite. Let **SA** be the category of strongly algebraic domains.

Theorem 14 $\mathbf{P} \subseteq \mathbf{SA} \subseteq \mathbf{Alg}$ *and none of the inclusions is reversible.*

In fact, **SA** has quite a lot of objects that **P** does not have. In particular, $\mathbf{Set} \subseteq \mathbf{SA}$. However, it follows from Theorem 13 that **SA** is not closed under function spaces. But this flaw may be partially remedied by changing the arrows on the category to allow *partiality.*

Definition: Let D and E be dcpo's. A partial function $\phi : D \rightharpoonup E$ is continuous iff it is defined on an open subset of D and preserves directed lub's. If $\psi : D \rightharpoonup E$ then say $\phi \sqsubseteq \psi$ iff for every $x \in D$, $\phi(x)\downarrow$ implies $\psi(x)\downarrow$ and $\phi(x) \sqsubseteq \psi(x)$. Let $[D \rightharpoonup E]$ be the poset of continuous partial functions from D to E.

For any category $\mathbf{C}$ of dcpo's, let $\mathbf{C}^\partial$ be the category having the same objects as $\mathbf{C}$ and continuous partial functions as arrows. Note that: $\mathbf{Set}^\partial \subseteq \mathbf{SA}^\partial$. We have the following:

Theorem 15 *If D and E are dcpo's then $[D \rightharpoonup E]$ is a dcpo and if D and E are strongly algebraic then $[D \rightharpoonup E]$ is strongly algebraic.*

While $\mathbf{SA}^{\partial}$ still fails to be cartesian closed, it does have nice categorical properties closely resembling cartesian closure (In [12], for example, such categories are called *partial cartesian closed.*) It is easy to show that $D + E$ and $D \times E$ are strongly algebraic whenever D and E are. Although $+$ is still a categorical coproduct even with the new partial functions $\times$ is *not* a categorical product on $\mathbf{SA}^{\partial}$. Actually, there *is* a product on $\mathbf{SA}^{\partial}$ given by the operator $A \times\times B = A + (A \times B) + B$ but $\times\times$ is less important than $\times$. We now show that $\mathbf{SA}^{\partial}$ satisfies Condition 2 quite nicely.

Definition: A continuous partial function $\phi : E \rightharpoonup D$ is a *projection* if there is a continuous partial function $\psi : D \rightharpoonup E$ such that $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi \sqsubseteq \mathrm{id}$.

Let $\mathbf{C}^{\partial P}$ have the obvious meaning. Then we have the following:

Theorem 16 *If $\mathbf{C}^{\partial P}$ has inverse limits and $F : \mathbf{C}^{\partial P} \to \mathbf{C}^{\partial P}$ is continuous then there is a D such that $D \cong F(D)$.*

Now, except for $\to$, all of the operators so far defined (including the partial funtion space) can be made into continuous functors on $\mathbf{DCPO}^{\partial P}$ and $\mathbf{SA}^{\partial P}$. Moreover, both of these categories have inverse limits. Since the empty set 0 is a terminal object in both categories, we can find fixed points for *all* of the operators (except those involving the total function space). So the sacrifice of cartesian closure offers a nice return on fixed point existence for data type specification. Moreover, when the meaning of a program construct is being specified via a fixed point equation, this is typically done by getting a fixed point for a total function $\gamma : [D \rightharpoonup D] \to [D \rightharpoonup D]$. But for every D, the space $[D \rightharpoonup D]$ of continuous partial functions has a least element—namely the totally undefined function. Thus, γ has a canonical fixed point. So Condition 2 is taken well in hand by $\mathbf{SA}^{\partial}$

$\mathbf{SA}^{\partial}$ has one more pleasing feature. Namely, it satisfies a "Smyth-like" theorem with respect to the algebraics:

Theorem 17 *If D is algebraic and $[D \rightharpoonup D]$ is algebraic then D is a strongly algebraic.*

Proof of the theorem is quite easily obtained by making a few (quite minor) additions to the standard proof of Theorem 10. The trick is to use the notion of a *strict function* between pointed domains. A continuous function $f : D_{\perp} \to E_{\perp}$ is *strict* if $f(\perp) = \perp$. It is not hard to show that $\mathbf{Alg}^{\partial}$ is isomorphic to the category $\mathbf{Alg}_{\perp}^{strict}$ of pointed algebraic dcpo's with strict functions and it is easy to see how to carry out Smyth's argument in this latter category. But this is really not a very efficient method of proof. The following question is motivated: *Is there are proof of Theorem 17 from Smyth's Theorem?*

There is also a nice topological link between the profinite domains and the strongly algebraic domains:

Theorem 18 *A strongly algebraic domain D is profinite iff the Scott topology on D is compact.*

A definability result for **BCAlg**. If one thinks of Smyth's theorem as saying that $\mathbf{P}_\perp$ is the largest subcategory of **Alg** having an interesting property P then one can ask of some of the other categories we have discussed whether they may likewise be distinguished by an picking an appropriate property P. There is a notion from logic which does this for **BCAlg**. We make the following:

Definition: Let us say that a class $\mathcal{K}$ of algebraic dcpo's is *definable* if there is a first order theory T in the language of posets such that

1. T extends the theory of posets, and
2. for any algebraic cpo D, $D \in \mathcal{K}$ iff D^0 is countable and $D^0 \models T$.

Let us show that **BCAlg** is definable. For each $n > 1$, let $\mathrm{UB}_n(\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_n)$ be the formula $\mathbf{v}_1 \preceq \mathbf{u} \wedge \ldots \wedge \mathbf{v}_n \preceq \mathbf{u}$ where $\preceq$ is a binary relation symbol. Now, consider the theory T generated by the universal closure of the following axiom scheme:

$$(\exists \mathbf{u}.\ \mathrm{UB}_n(\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_n)) \rightarrow \exists \mathbf{u}.\ \mathrm{UB}_n(\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_n) \wedge (\forall \mathbf{w}.\ \mathrm{UB}_n(\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_n) \rightarrow \mathbf{u} \preceq \mathbf{w})$$

together with the axioms which assert that $\preceq$ is reflexive, anti-symmetric and transitive. The models of T are exactly the bounded complete posets. One can show that an ω-algebraic dcpo is bounded complete iff its basis is countable and bounded complete. It follows therefore that **BCAlg** is definable. One may use the Compactness Theorem for first order logic, together with Theorem 13 to show that, in fact, the following holds:

Theorem* 19 *The bounded complete algebraic dcpo's are the largest cartesian closed definable category.*

4. Continuous dcpo's.

There is a Condition 5 problem with all of the categories mentioned so far if one wishes to have the *real number data type*. The problem is that the extended reals are *not* algebraic! The main difficulty in dealing with this is to get a more general class of dcpo's which satisfy Condition 3. To accomplish this we use the following:

Definition: Let D be a dcpo. For $x, y \in D$, $y \ll x$ if for every directed subset $M \subseteq D$, $\bigsqcup M \sqsupseteq x$ implies $z \sqsupseteq y$ for some $z \in M$. D is (ω-)*continuous* if there is a countable set $B \subseteq D$ called a *basis* for D such that for each $x \in D$, the set $\hat{x} = \{y \in B \mid y \ll x\}$ is directed and $x = \bigsqcup \hat{x}$.

The idea of a continuous *lattice* is due to Scott [18] and they have been studied in quite a lot of detail [3]. Continuous dcpo's have received less attention because they are less tractable and

ALGEBRAIC	*CONTINUOUS*
Alg	Continuous dcpo's
P	**RP**
$\mathbf{P}_\perp$	$\mathbf{RP}_\perp$
BCAlg	Bounded complete continuous dcpo's
CohAlg	Coherent continuous dcpo's
AlgLat	Continuous Lattices

Table 1: Categories of retracts.

because most of the continuous dcpo's that come up in mathematics are lattices. But as far as domain theory goes, the more general notion is useful. The main point is this: it is possible to define a quite satisfactory notion of computability for (ω-)continuous dcpo's and continuous functions between them. See [32] for a nice exposition. Moreover, this theory generalizes the computability theory for algebraic dcpo's. For the rest of this section we assume that all of the continuous dcpo's are countably based. There is a close relationship between algebraic and continuous dcpo'given by the following:

Theorem 20 *A dcpo D is continuous if and only if there is a algebraic dcpo E and retraction $r : E \to D$.*

Like the algebraics, the continuous dcpo's fail to form a ccc. It is easy to find cartesian closed subcategories, however, because of the following:

Theorem 21 *If* **C** *is cartesian closed then the category* **RC** *of retracts of objects of* **C** *is also cartesian closed. Moreover, the category* **PC** *of projects of objects of* **C** *is cartesian closed.*

Table 1 lists some of what this tells us. The column on the right lists the category that one gets from the corresponding category on its left through taking retracts. Except for the continuous dcpo's, all of the categories listed in the "continuous" column are ccc's. Note how simply the classes of retracts of objects in **BCAlg, CohAlg,** and **AlgLat** can be described. Relatively little is known about this retracts correspondence in general beyond facts that come out of the proof of Theorem 21. For example: *If* $\mathbf{C}^P$ *has inverse limits, does* $\mathbf{RC}^P$ *also have inverse limits? What about* $\mathbf{PC}^P$*? When does* $\mathbf{RC} = \mathbf{PC}$*?*

Properties of **RP.** Some things are known about the specific categories mentioned in Table 1. It is well-known that the last three categories in the "continuous" column have inverse limits and that the retracts and projects categories are the same. Gunter [4] has shown that it is possible to find fixed points for a large class of operators on **RP**, but these results fall shy of showing that $\mathbf{RP}^P$ has inverse limits. It has long been an unsolved problem to characterize the retracts of profinites *intrinsically.* The following partial solution is due to Gordon Plotkin and Achim Jung:

Theorem* 22 *Let D be a dcpo. The following are equivalent:*

1. *D is a retract of a profinite domain.*

2. *D is a project of a profinite domain.*

3. *There is an ω-sequence of continuous functions $f_i : D \to D$ such that for each $i, j \in \omega$,*

   (a) $i \leq j$ implies $f_i \sqsubseteq f_j$,

   (b) $im(f_i)$ is finite,

   (c) $\bigsqcup_{i \in \omega} f_i = \mathrm{id}_D$.

The theorem has two interesting consequences:

Theorem* 23 $\mathbf{P} = \mathbf{Alg} \cap \mathbf{RP}$.

Theorem 24 *Suppose $D \in \mathbf{RP}$. Then D is profinite if and only if the Scott topology on D has a basis of compact open sets.*

Another intrinsic characterization which concentrates more on the basis of the domain was derived independently by Kamimura and Tang [6].

We end by mentioning two more unanswered questions about **RP**. Scott [18] showed a correspondence between the continuous lattices and the T_0-injectives. A similar result characterizes the bounded complete continuous dcpo's as corresponding to a natural class of topological spaces. It seems reasonable to ask: *is there some way of characterizing the retracts of profinites topologically?* A second question is whether there is a "Smyth-like" theorem for **RP**. This is Tang's Conjecture: *If $[D \to D]$ is continuous then D is a retract of a profinite domain.*

Appendix.

Proofs of Theorems in Section 3.

Proof of Theorem 9: We need the following isomorphisms:

1. $D \times (E + F) \cong (D \times E) + (D \times F)$,
2. $[(D + E) \to F] \cong [D \to F] \times [E \to F]$.

These hold for any D, E, F. If, moreover, D has a least element then we also have the following:

3. $[D \to (E + F)] \cong [D \to E] + [D \to F]$.

Proofs of 1 and 2 are left for the reader. To see 3, suppose $f : D \to (E + F)$ is continuous. If $f(\bot) \in E$ then $im(f) \subseteq E$, for if $x \in D$ then $\bot \sqsubseteq x$ so $f(\bot) \sqsubseteq f(x)$. Similarly, $f(\bot) \in F$ implies $im(f) \subseteq F$. Now, if $f(\bot) \in E$ then define $\hat{f} : D \to E$ to be the corestriction of f to E. This makes sense because $im(f) \subseteq E$. Also, $\hat{f}$ is obviously continuous. If, on the other hand, $f(\bot) \in F$ then take $\hat{f} : D \to F$ to be the corestriction of f to F. It is not hard to see that the correspondence $f \mapsto \hat{f}$ is an isomorphism between $[D \to (E + F)]$ and $[D \to E] + [D \to F]$.

It follows immediately from Isomorphism 1 that $\mathbf{C}'$ is closed under products. To see that it is also closed under ther function space operation, suppose S and T are in $\mathbf{C}'$. To save ourselves some subscripts, let's just assume that $S = D + E$ and $T = F + G$ where D, E, F, G are in $\mathbf{C}$ (the calculations go through equally well if S or T has any finite number of such disjoint components). We compute as follows:

$$\begin{aligned}[(D + E) \to (F + G)] &\cong [D \to (F + G)] \times [E \to (F + G)] && \text{by Iso 2} \\ &\cong ([D \to F] + [D \to G]) \times ([E \to F] + [E \to G]) && \text{by Iso 3}\end{aligned}$$

since D and E have least elements. After using Isomorphism 1 to pull the $+$'s to the outside of the expression the result has the form of an object in $\mathbf{C}'$. Finally, if $\mathbf{C} \subseteq \mathbf{Alg}_\bot$ then $\mathbf{C}' \subseteq \mathbf{Alg}$ so by Theorem 13, $\mathbf{C}' \subseteq \mathbf{P}$.

Proof of Theorem 12: We may obtain the Theorem as a corollary of Smyth's Theorem together with the following:

Lemma 25 *If D is a pointed dcpo and $[D \to D]$ is algebraic, then D is algebraic.*

Proof. Suppose $f : D \to D$ is finite (as an element of $[D \to D]$). We claim that $f(\bot)$ is finite. Suppose $a_0 \sqsubseteq a_1 \sqsubseteq \cdots$ is a chain in D with $f(\bot) = \bigsqcup_n a_n$. For each n, define $f_n : D \to D$ by

$$f_n(x) = \begin{cases} f(x) & \text{if } x \neq \bot; \\ a_n & \text{if } x = \bot. \end{cases}$$

These functions are all continuous and $\bigsqcup_n f_n = f$ so $f = f_n$ for some n. Hence $f(\bot) = f_n(\bot) = a_n$ and $f(\bot)$ must therefore be finite. Now, suppose $d \in D$ and let $f(x) = d$ be the constant function determined by d. Since $[D \to D]$ is ω-algebraic, there are finite functions $f_0 \sqsubseteq f_1 \sqsubseteq \cdots$ such that $f = \bigsqcup_n f_n$. Hence $\bigsqcup_n f_n(\bot) = f(\bot) = d$. But $f_n(\bot)$ is finite for each n so D must be algebraic.

Proof of Theorem 19: Let A be a poset and suppose $u \subseteq A$. An upper bound x for u is said to be *minimal* if whenever $y \sqsubseteq z$ for every $y \in u$ and $z \sqsubseteq x$ then $x = z$. Let

$$\mathrm{MUB}_A(u) = \{x \mid x \text{ is a minimal upper bound for } u \text{ in } A\}.$$

We will need the following fact about profinite domains (see [4] or [15]): if D is profinite and $u \subseteq D^0$ is bounded then $\mathrm{MUB}_{D^0}(u)$ is *finite and non-empty.* Let us call this "property $(*)$".

We work in a language with a single binary relation symbol $\preceq$ Let T be a first order theory extending the theory of posets. Suppose, moreover, that if A is a model of T then $A \times A$ is a model of T and that every model of T has property $(*)$. Let A be a model of T in which the interpretation of $\preceq$ is not bounded complete. Then there is a finite (possibly empty) set $u \subseteq A$ such that $\mathrm{MUB}(u)$ has at least two elements. Suppose u has n elements. For each integer $m \geq 2$ we show that there is a model of T satisfying the first order axiom

$$\phi_m \equiv \exists \mathbf{v}_1 \cdots \exists \mathbf{v}_m . \bigwedge_{i \neq j} \mathbf{v}_i \neq \mathbf{v}_j \wedge \mathbf{v}_i \in \mathrm{MUB}(\{\mathbf{c}_1, \ldots, \mathbf{c}_n\})$$

for constants $\mathbf{c}_1, \ldots, \mathbf{c}_n$ not contained in $\mathcal{L}$. Note that A is a model of ϕ_2 if $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are interpreted by the elements of u. So suppose we know that $T \cup \{\phi_m\}$ has a model B in which $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are interpreted by $X_1, \ldots, X_n$. We claim that $B \times B$ is a model of $T \cup \{\phi_{m+1}\}$ when $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are interpreted by $(X_1, X_1), \ldots, (X_n, X_n)$. To see this, let $v = \{X_1, \ldots, X_n\}$ and $w = \{(X_1, X_1), \ldots, (X_n, X_n)\}$. Then

$$\mathrm{MUB}(w) = \mathrm{MUB}(v) \times \mathrm{MUB}(v).$$

But there are m^2 elements in $\mathrm{MUB}(v) \times \mathrm{MUB}(v)$. Since $m^2 > m + 1$ for $m > 1$ we are done. Now, for each m, $\phi_{m+1} \to \phi_m$ so we may deduce that any finite subset of $T \cup \{\phi_m \mid m \geq 2\}$ has a model. Hence, by the Compactness Theorem, there is a model C of $T \cup \{\phi_m \mid m \geq 2\}$. But if C interprets $\mathbf{c}_1, \ldots, \mathbf{c}_n$ by $Y_1, \ldots, Y_n$ respectively then C cannot have property $(*)$ because $\mathrm{MUB}(\{Y_1, \ldots, Y_n\})$ must be infinite. But this contradicts our assumption that models of T satisfy $(*)$. We conclude that all of the models of T must be bounded complete.

We are therefore able to conclude that the bounded complete dcpo's are the largest definable subcategory of the profinites which is closed under products. But by Theorem 13, any cartesian closed category subcategory of the algebraics must be a subcategory of the profinites. The desired conslusion therefore follows.

Proofs of Theorems in Section 3.

Proof of Theorem 22: Since a projection is a retraction we certainly have (2) $\Rightarrow$ (1). We show that (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

(1) $\Rightarrow$ (3). Suppose E is ω-profinite and there are continuous functions $r : E \to D$ and $r' : D \to E$ such that $r \circ r' = \mathrm{id}_D$. Since E is ω-profinite, there is a sequence $\langle p_i \rangle_{i \in \omega}$ of continuous idempotents on E such that $im(p_i)$ is finite for each i, $p_i \sqsubseteq p_j$ whenever $i \leq j$ and $\bigsqcup_i p_i = \mathrm{id}_E$. For each $i \in \omega$, define a continuous function $f_i = r \circ p_i \circ r' : D \to D$. If $i \leq j$ then $f_i = r \circ p_i \circ r' = r \circ p_j \circ r' = f_j$. Moreover, $\bigsqcup_i f_i = \bigsqcup_i r \circ p_i \circ r' = r \circ (\bigsqcup_i p_i) \circ r' = r \circ r' = \mathrm{id}_D$. Finally, $im(f_i)$ is finite for each i because $im(p_i)$ is. Thus the sequence $\langle f_i \rangle_{i \in \omega}$ satisfies (a), (b) and (c).

(3) $\Rightarrow$ (2). Suppose D is a dcpo and $\langle f_i \rangle_{i\in\omega}$ is a sequence of functions satisfying conditions (a), (b) and (c). Let E be the set of *monotone* sequences $x : \omega \to D$ such that for each $i \in \omega$, $x_i \in F_i = \bigcup_{j\leq i} im(f_j)$ and

$$f_i(\bigsqcup_{j\in\omega} x_j) \sqsubseteq x_i. \tag{$*$}$$

Order E coordinatewise, *i.e.* $x \sqsubseteq y$ if and only if $x_i \sqsubseteq y_i$ for each i. We claim that E is profinite. To see that E is a dcpo, suppose $M \subseteq E$ is directed. We show that the least upper bound x of M in $\prod_{i\in\omega} F_i$ is in E. Now, x is certainly monotone; to prove that x satisfies condition $(*)$, we calculate

$$\begin{aligned}
f_i(\bigsqcup_{j\in\omega} x_j) &= f_i(\bigsqcup_{j\in\omega} \bigsqcup\{y_j \mid y \in M\}) \\
&= f(\bigsqcup\{\bigsqcup_{j\in\omega} y_j \mid y \in M\}) \\
&= \bigsqcup\{f_i(\bigsqcup_{j\in\omega} y_j) \mid y \in M\} \\
&\sqsubseteq \bigsqcup\{y_i \mid y \in M\} && \text{by } (*) \text{ for } y \in M \\
&= x_i.
\end{aligned}$$

To see that E is profinite, define for each $n \in \omega$ a function $p_n : E \to E$ by letting $p_n(x) = y$ where

$$y_i = \begin{cases} x_i & \text{if } i \leq n; \\ x_n & \text{otherwise.} \end{cases}$$

The following conditions are satisfied for each n:

- p_n is continuous,
- $p_n = p_n \circ p_n \sqsubseteq \mathrm{id}_D$,
- $im(p_n)$ is finite, and
- $p_n \sqsubseteq p_m$ for $n \leq m$.

Since we also have $\bigsqcup_{n\in\omega} p_n = \mathrm{id}_D$, it follows that E is profinite. To complete the proof, define $p : E \to D$ by $p : x \mapsto \bigsqcup_{j\in\omega} x_j$ and $q : D \to E$ by $q : x \mapsto \langle f_i(x) \rangle_{i\in\omega}$. It is easy to check that p and q are continuous. If $x \in D$ then $(p \circ q)(x) = \bigsqcup_{i\in\omega} f_i(x) = x$. If $x \in E$ then $(q \circ p)(x) = q(\bigsqcup_{j\in\omega} x_j) = \langle f_i(\bigsqcup_{j\in\omega} x_j) \rangle_{i\in\omega} \sqsubseteq \langle x_i \rangle_{i\in\omega}$. Hence D is the continuous projection of a countably based profinite domain.

Proof of Theorem 23: If a domain is profinite then it is algebraic and it is a retract of itself, so to prove the theorem we must show that a retract of a profinite domain which is algebraic is profinite. The proof uses the characterizations of **P** and **RP** given by Theorems 5 and 22(3) respectively.

Suppose $f : D \to D$ is a continuous function with a finite image such that $f(x) \sqsubseteq x$ for each x. Then for any n and any x, $f^{n+1}(x) \sqsubseteq f^n(x)$. Since f has a finite image it follows that for some m, $f^{m+1}(x) = f^m(x)$. So define $f_\infty : D \to D$ by setting $f_\infty(x) = f^m(x)$ where $f^{m+1}(x) = f^m(x)$. This function is monotone, for if $x \sqsubseteq y$, $f^m(x) = f_\infty(x)$ and $f^n(y) = f_\infty(y)$ then for any $l \geq m, n$ we have $f_\infty(x) = f^l(x) \sqsubseteq f^l(y) = f_\infty(y)$. Since the image of f_∞ is finite, it follows that f_∞ is continuous. Moreover, if $x \in D$ and $f^{n+1}(x) = f^n(x)$ then $f_\infty^2(x) = f^{2n}(x) = f^n(x) = f_\infty(x)$ so f_∞ is idempotent. The set $M_\infty = \{f_\infty \mid f \in M\}$ is directed so there is a continuous function $g = \bigsqcup M_\infty$. We claim that g is the identity map on D. To see this, suppose $e \in D^0$. Now $e = (\bigsqcup M)(e)$ so $e \sqsubseteq f(e)$ for some $f \in M$. Hence $f(e) = e$. But this means $f^n(e) = e$ for all n so apparently $f_\infty(e) = e$. Thus $g(e) = e$ and since D is algebraic we conclude that g is the identity function.

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