PROFINITE SOLUTIONS FOR RECURSIVE DOMAIN EQUATIONS

Carl Gunter

Carnegie-Mellon University
Department of Computer Science
Pittsburgh, Pennsylvania 15213

Preface

This dissertation arose out of a fateful conversation that I had with Jon Barwise in August of 1982. I wanted to write a thesis on a topic in semantics so he suggested that I read Dana Scott's 1982 **ICALP** paper. I found the subject of that paper very interesting and followed it up by reading almost everything I could find on the theory of domains. When Professor Barwise accepted the directorship of CSLI in the Spring of 1983, he spoke to Scott about my work and I was offered an opportunity to come to CMU and write my dissertation under Professor Scott's direction.

I arrived at CMU in June of that year and found it to be a wonderful atmosphere for the continued study. First and foremost I benefited from Professor Scott's diligent and direct supervision. The breadth and depth of his contributions to mathematical logic, philosophy, semantics and many other areas has been an inspiration to me for the few years since I read that paper which Barwise gave me. For his assistance with my own research, I owe a lasting debt of gratitude. Moreover, I was a bit more of a problem for him than many graduate students because I was writing my dissertation for the University of Wisconsin while visiting at CMU. But somehow, Professor Scott was able to find support for me which allowed me ample opportunity for research. The result, some 18 months since arriving at CMU, is this dissertation.

My wife, Elsa, has been a tremendous help to me throughout my years in graduate school and has displayed an especially great degree of concern and forbearance in the last two years. She offered not only invaluable emotional support and encouragement, but also technical assistance on typesetting and proofreading. Some of her typesetting talents are evidenced by Figures 2.1 and 6.1 and the cross-referencing system that created the tables of categories. Moreover, she assisted with the proofs of some of the theorems in Chapter 4. Much of this was done at the expense of her own graduate work at the University of Wisconsin.

Many people at CMU assisted me with the dissertation. Steve Brookes, Ana Pasztor and Rick Statman on the CMU faculty have been especially helpful. Glynn Winskel was also very helpful to me in my first few months at CMU. Among the CMU graduate students, Paola Gianinni provided timely encouragement and was a patient listener. Achim Jung did careful proofreading of the early chapters and made several noteworthy technical contributions such as an essential idea in the proof of Theorem 5.7. Pino Rosolini helped me with partial functions and proofread my Chapter 7. I owe thanks for encouragement to Andrew Weiss and Anne Rogers. Anne also introduced me to the CMU computing facilities and looked over some of my most horrible Hasse diagrams.

Gordon Plotkin provided assistance and advice. As anyone who reads the thesis will see,

he also provided a considerable degree of inspiration. I had several helpful conversations with Adrian Tang and Tsutomu Kamimura. As for people at the University of Wisconsin, I would certainly like to thank Jon Barwise for getting me started working on the theory of domains. I thank also Kenneth Kunen (who was himself a student of Professor Scott) for agreeing to be my University of Wisconsin supervisor. Thanks are also owed to my other teachers there; these include Jerry Keisler, Terry Millar, and Mike Byrd. I also learned from many of the graduate students at Wisconsin. Mark Bickford, Ali Enayat, Darrah Chavey, Sergio Fajardo, and Charlie Mills deserve special mention, although there are many others.

Contents

List of Figures

Abstract

The purpose of the dissertation is to introduce and study the category of profinite domains. The study emphasizes those properties which are relevant to the use of these domains in a semantic theory, particularly the denotational semantics of computer programming languages. An attempt is made to show that the profinites are an especially natural and, in a sense, *inevitable* class of spaces. It is shown, for example, that there is a rigorous sense in which the countably based profinites are the largest category of countably based spaces closed under the function space operation. They are closely related to other categories which appear in the domain theory literature, particularly strongly algebraic domains (**SFP**) which form a significant subcategory of the profinites. The profinites are *bicartesian closed*—a noteworthy property not possessed by **SFP** (because it has no coproduct). This gives rise to a rich type structure on the profinites which makes them a pleasing category of semantic domains.

However, there are problems that arise with respect to the solution of recursive domain equations over the profinites which do not apply to the strongly algebraic domains. For the purposes of semantics, the solution of such equations is essential because it is the primary technique of data type specification. There are continuous functors over the profinites which have *no* profinite solution. The usual universal domain techinique for solutions to such equations will not work for the profinites because there is no universal profinite domain. Instead a kind of "multi-universal domain" technique is devised which uses an infinite class of "almost universal" spaces. These make it possible to show that an equation of the form $D \cong F(D)$ where F is a locally continuous endofunctor on profinites has a solution if and only if a related equation has a finite solution. For a continuous computable functor the decision problem for the existence of a fixed point is Σ_1. The existence result is also used to prove properties of solutions. For example, it is shown that a countably based fixed point of the diagonal of the function space operation must have a least element.

Introduction

The purpose of the dissertation is to introduce and study the category of profinite domains and some related categories of partial orders. We discuss some of the methods for obtaining solutions to domain equations, especially solutions which must satisfy particular conditions such as profiniteness. A secondary theme is the elegant and natural description of relevant categories and functors. There are seven chapters; the first four are about categories of domains and their properties. The next two discuss how these properties are used to solve equations and the last chapter is about partial functions and pre-domains.

More specifically, Chapter 1 discusses motivation, history and some basic category theory. Chapter 2 presents the primary intrinsically characterized classes of partial orders which will be our objects of study. The realtionship between pre-orders and algebraic cpo's is examined in some detail. The category of Plotkin orders is introduced; we display exponential and product functors for this category and show that it is cartesian closed. Chapter 3 discusses the notion of an adjunction (or galois connection) between cpo's and looks at the cpo of algebraic deflations. We show the existence of inverse limits in the category of cpo's with upper adjoints as morhpisms and provide useful conditions under which a class of cpo's will have such limits. Chapter 4 looks first at properties which distinguish classes of algebraic cpo's in terms of properties such as cartesian closure and first order definability. We also discuss some intrinsically characterized cartesian closed sub-categories of profinite domains. The last two sections of Chapter 4 discuss the Scott and Lawson topologies on profinite domains and some related classes. Chapter 5 presents universal domains for various profinite categories in a way which emphasizes the *model theoretic* significance of their constructions. It is shown how these universal domains can be used to obtain solutions for domain equations. Chapter 6 introduces some noteworthy functors and their closure properties with respect to the category of profinite domains. A necessary and sufficient condition for the solvability of equations involving continuous functors is given and some of the consequences of this characterization for particular functors is explored. In Chapter 7 we attempt to motivate the need for a theory of partial functions at higher types and discuss how cpo's arise naturally in this way. We define the category of pre-domains and give a treatment of this class which is similar to that given to profinite domains in the earlier chapters.

1. Methodology and basic definitions. The categories which we introduce are examined from the point of view of their potential applications as semantic domains in, for example, the denotational semantics of computer programming languages. However, a variety of mathematical properties having no apparent immediate application are also investigated. The study is carried out in the spirit of mathematical investigations in the theory of domains such as those studies given to continuous lattices, information systems and many other related classes. The results we prove are meant to demonstrate that the profinite domains form an elegant, simple, natural and, in a sense, *inevitable* category of spaces. On the profinites are defined a host of interesting functors whose fixed point properties are non-trivial. However, a variety of techniques for finding and reasoning about such fixed points are demonstrated.

The profinites arise quite naturally from category-theoretic considerations when one accepts as interesting certain categories of partial orders which consitute the mathematical foundation of the Scott-Strachey theory of programming semantics. They are an especially "large" collection of semantic domains which contain many of the categories of domains previously studied for the purposes of programming semantics. Indeed, there is a rigorous sense in which the profinites are the *largest* category having certain relevant properties. Although the dissertation does not discuss the topic in any detail, the profinites can also be given a satisfactory computability theory in keeping with the well-known treatments of *effectively given domains.*

Define a complete poset (cpo) to be a poset in which every directed subset M has a least upper bound $\bigsqcup M$. A function between cpo's is *(Scott) continuous* if it is monotone and preserves such lub's. If D and E are cpo's then, with the pointwise ordering, the poset $[D \to E]$ of continuous functions from D into E forms a cpo. Let D be a cpo and let M_D be the set of continuous functions $p : D \to D$ such that $p = p \circ p \sqsubseteq \mathrm{id}_D$ and the image of p is finite. Then D is said to be *profinite* if M_D is directed and $\bigsqcup M_D = \mathrm{id}_D$. If M_D is countable then D is said to be ω-profinite. This is one of several equivalent conditions which can be used to define profinite domains.

Historically a number of categories closely related to the profinites have been studied. The best known is the category **SFP** which is a significant sub-category of the profinites. The names "**SFP**" and "profinite" arise from category-theoretic considerations which the dissertation discusses in some detail. A cpo is said to be *algebriac* if it has a basis of finite (compact) elements and ω-algebraic if that basis is countable. We show that profinite domains are algebraic and give a quite easy to use characterization of the kinds of posets which can arise as the bases of profinite domains. These are called *Plotkin posets* and we use them to develop a kind of "information systems" approach to the profinite domains.

2. Functors. It is shown in the dissertation that the profinite domains have finite products and coproducts as well as terminal and initial objects. Moreover, they are closed under the (continuous) function space operation and form a *bicartesian closed category.* This gives rise to a very rich type structure which is a primary topic of study. In particular, one would like to know exactly when a domain equation (such as $D \cong [D \to D]$ or $E \cong E \times E$) has a profinite solution. The profinite solution of such equations can be problematic. For example, the equation $D \cong 1 + [D \to D]$ (where 1 is the singleton cpo and + is the coproduct) has *no* profinite solution. A fairly satisfying necessary and sufficient condition for when a continuous endofunctor F on the profinites has such a solution is derived in the dissertation. It is shown that if D is profinite then the set M_D has a least element p whose image is a finite poset called the *root* of D. The functor F has a fixed point if and only if there is a poset A such that A is isomorphic to the root of $F(A)$. This says that the original equation invloving F can be solved exactly when a related equation has a finite solution. For computable functors this shows that the decision problem for existence of a fixed point is Σ_1.

The existence of fixed points of functors is generally proved by the use of category theory or by the use of a *universal domain*. Since no universal domain for the profinites exists, it is necessary to derive a "multi-universal domain" technique involving an infinite class of domains which have some of the properties of the well-known examples of universal domains. Formally, the following result is proved:

> **Theorem:** *Let A be a finite poset which is equal to its own root. Then there is a poset $A^\star$ such that for every ω-profinite poset D with root isomorphic to A, there is a function $p : A^\star \to A^\star$ such that $p = p \circ p \sqsubseteq \mathrm{id}_{A^\star}$ and D is isomorphic to the image of p.*

The construction of these domains is carried out in some detail and we stress the theme that the universality of the structures arises from the fact that they are *saturated* in the model theoretic sense. They are used to give necessary and sufficient conditions for the existence of fixed points for locally continuous functors over the category of *retracts* of profinites.

There are a great many functors defined on the profinite domains. We show that a continuous functor on cpo's which sends finite posets to finite posets cuts down to an endofunctor on the profinites. On the profinites we define, for example, three *powerdomains* which are analogous to the well-known examples of such functors. A number of other noteworthy functors are also studied. For an algebraic cpo D, let $G(D)$ be the poset of continuous functions $p : D \to D$ such that $p = p \circ p \sqsubseteq \mathrm{id}_D$ and $im(p)$ is an algebraic cpo. We show that if D is profinite then $G(D)$ is an algebraic cpo which has a locally finite basis, *i.e.* between any two elements of the basis there are only finitely many basis elements. Another interesting functor is the *join completion* $\mathcal{J}(D)$. Say a cpo D is *bounded complete* if it is non-empty and each of its bounded finite subsets has a least upper bound. The join completion is defined on algebraic cpo's and $\mathcal{J}(D)$ is bounded complete for every D. Moreover, an algebraic cpo is a fixed point of $\mathcal{J}$ if and only if it is bounded complete. This functor can be used to get a universal bounded complete poset which is not isomorphic to Scott's well-known universal domain for this class.

3. Limits and duality. Let D and E be cpo's and suppose $p : E \to D$ and $q : D \to E$ are monotone. If $p \circ q \sqsupseteq \mathrm{id}_D$ and $q \circ p \sqsubseteq \mathrm{id}_E$ then let us call p an *upper adjoint* and q a *lower adjoint*. Let $\mathbf{CPO}^\uparrow$ be the category of cpo's and continuous upper adjoints. We prove the following

> **Theorem:** *A cpo is profinite if and only if it is the limit in $\mathbf{CPO}^\uparrow$ of an inverse system of finite posets. Moreover, if Δ is an inverse system in the category $\mathbf{P}^\uparrow$ of profinite domains and continuous upper adjoints then its limit in that category exists and coincides with its limit in the category of cpo's and (Scott) continuous functions.*

A set of simple conditions for proving results like this for categories of algebraic cpo's is provided and applied to several examples. To mention one of these, define a category $\mathbf{C}$ as follows. The objects of $\mathbf{C}$ are continuous cpo's D such that the Scott compact open subsets of D form a basis (for the Scott topology) and this basis is closed under finite intersections. We show that when the arrows of $\mathbf{C}$ are continuous upper adjoints, $\mathbf{C}$ has limits for inverse systems. The limit existence results give rise to a rather general form of limit/colimit duality for $\mathbf{P}^{\uparrow}$ and the dual category of profinites with lower adjoints. We also demonstrate the continuity of the function space functor on $\mathbf{P}^{\uparrow}$.

4. Closure properties. Although many functors on cpo's send profinites to profinites, there are natural ones which do not. For example, if D is a cpo, let $\prod_\omega D$ be the product of countably many copies of D. By analyzing roots of products one can see that $\prod_\omega D$ is profinite if and only if D is empty or has a least element. Since any finite poset is profinite and there are finite posets without least elements, it follows that there are profinite domains D for which $\prod_\omega D$ is not profinite. However, the most interesting functor—the function space—*does* send profinites to profinites. But something more is true:

> **Theorem:** *If D and $[D \to D]$ are ω-algebraic cpo's then D is profinite.*

The proof is similar to that of Smyth's theorem (which is the case where the posets have least elements). The theorem can be used to obtain the following surprising corollary: if D is a non-empty ω-algebraic cpo and $D \cong [D \to D]$ then D has a least element! Hence the study of cpo's D such that $D \cong [D \to D]$ naturally leads to consideration of the profiniteness condition (and, it appears, the least element). Other results like the above theorem are also demonstrated, including a slight generalization of Smyth's theorem: if D is a *cpo* with a least element and $[D \to D]$ is ω-algebraic then D is profinite.

Let $\mathcal{K}$ be a class of ω-algebriac cpo's and let $\mathcal{K}_0$ be the class of posets A such that A is isomorphic to the poset of finite elements of a member of $\mathcal{K}$. Say $\mathcal{K}$ is *elementary* if $\mathcal{K}_0$ is the class of countable models of a first order theory. We show that the largest class $\mathcal{K}$ of ω-algebraic cpo's which is elementary and cartesian closed is the class of bounded complete ω-algebraic cpo's. We also provide intrinsic characterizations of several sub-categories of the profinites which show that the profinites have many interesting cartesian closed sub-categories.

When one carries out the definiton of a category of limits of finite posets in the way described above for profinite domains but using partial rather than total functions then one arrives at a new category of algebraic cpo's which are called *pre-domains*. This name derives from the fact that a pre-domain is a cpo D such that $D_\perp$ (= the result of adding a new least element $\perp$ to D) is profinite. We present an "information systems" method for characterizing pre-domains and Scott continuous partial functions defined on them. We show that the pre-domains form the largest partial cartesian closed full sub-category of the ω-algebraic cpo's and continuous partial maps.

5. Topological properties. The Scott continuous functions between a pair of cpo's are exactly the continuous functions in the general topological sense when the cpo's are endowed with the *Scott topology*. Let us say that a cpo is *continuous* if it is the retract of an algebraic cpo and say that a poset is *finitely* continuous if it is the retract of a profinite domain. It is shown that an algebraic cpo is a continuous cpo with a basis of compact open sets and an ω-profinite cpo is a finitely continuous cpo with a countable compact open basis. We provide a set of conditions on the Scott topology of a continuous cpo which is equivalent to profiniteness of the cpo and various simple topological conditions are shown to lead to natural classes of algebraic cpo's. We also investigate a refinement of the Scott topology on cpo's called the *Lawson* topology and show that on a profinite domain, the Lawson topology is compact Hausdorff and 0-dimensional as would be expected by analogy with the theory for algebraic lattices and **SFP**-objects.

Chapter 1

Background

The purpose of this chapter is to set the stage for subsequent chapters by reviewing certain motivations, historical background, and a few important categorical notions.

1.1 Some questions about the theory of domains

We begin by discussing answers to four questions that anyone would be inclined to ask when being introduced to the theory of domains for the first time.

Why is a mathematical semantics for programming needed? Reasoning about the properties of a program written in a modern high-level programming language typically involves a complex mixture of ordinary mathematical reasoning and a kind of low-level reasoning about the machine or compiler which implements the language. For the most part, mathematical forms of reasoning have a firm logical foundation and a well developed methodology of significant generality and precision. Programming languages, on the other hand, and the systems which implement them are often fraught with inconsistencies (or at least idiosyncrasies). Their design can also be misleading, since many of them possess hidden features which make it impossible to reason correctly about the program in the apparently intended ways. We cannot make a detailed case for these accusations here, but the case has been made forcefully and specifically elsewhere ([Brookes 1985] and [Meyer 1984] are good examples). Of course, such characteristics make it difficult to tell what a program is likely to do on its input. This difficulty is reflected in the time required to debug programs and the questionable extent to which most programs can be "proven correct".

If a computer program can be given a really clear mathematical *meaning*, it may then be possible to prove rigorously the necessary properties of the program. And if a computer *language* is given such a meaning, then it will be possible to prove properties of programs written in that language in a systematic way. On the one hand, a good mathematical metalanguage will improve our ability to *specify* properties of programs and, on the other hand, it will make it possible to establish that a given program meets its specifications. Moreover, a clear semantical approach will help with the specification of the languages themselves and will therefore allow for proofs that a compiler correctly implements a whole language. This may even allow us someday to devise ways of generating good compilers for languages directly from semantic descriptions. The development, therefore, of a semantic theory of programming is an essential step in the progress toward the controlled design of more reliable software. Hence, the case for the *need* for mathematical semantics rests on many practical considerations.

What should such a semantic theory look like? Certainly one feature that a good semantic theory should have is "machine independence". Brookes [1985] puts it the following way, where he also explains how this independence is to be obtained:

> Certainly we will ignore details which are dependent on implementation on particular compilers or machines. Instead of specifying that a program, if run on such-and-such a machine under the so-and-so operating system with version n of the Pascal compiler, will do something specific to the contents of that machine's memory, we will give an abstract formal semantics *independent* of machines but which could (at least in principle) be related to what actually will go on when programs run in the real world. We ignore space, time and coding tricks when we describe the semantics of the language. Our semantics will be founded upon a universal set of abstract mathematical structures and objects, such as (input-output) functions, in terms of which we will be able to explain the meanings of syntactic constructs without relying on implementation details. We will, nevertheless, be able to give sufficiently precise descriptions for would-be *implementors* of a language to be able to produce a *correct* implementation (compiler, interpreter, or whatever) with respect to our formal standard. Programmers will be able (if they so wish) to express and prove properties of their programs, by appealing to the semantic definitions to explain precisely the effects of their programs.

It has become conventional wisdom in discussing programming-language semantics to divide the possible approaches into three main classes: the *denotational*, the *operational*, and the *axiomatic*. Briefly, the axiomatic method codifies the meaning of programs into a set of rules for deriving properties of programs. The construction of complete sets of rules is not always straightforward. The operational method achieves some machine independence by putting foward abstract machines on which the programs of the language are implemented in the usual way. This is an important method, because the abstract machines can be defined cleanly and precisely without the many compromises of actual machines. Once the language has an operational definition, then implementations can be given for many real machines by implementing the abstract machine, which is often much easier to do than implementing a whole language. The drawback in the approach is that an operational semantics may still be difficult to reason about. The reader is referred to [Stoy 1977] or [Brookes 1985] for more details.

As for the denotational method, the idea is not really in conflict with either of the other two methods. From one point of view, we can use denotational models to justify rules of inference, and thus prove consistency of a set of rules (which may not always be obvious). From another point of view, the denotational definition can be considered as a higher-level abstract implementation of the language. Operational definitions usually are concerned with manipulations of finite, discrete objects representing certain features of the state of the abstract machine. Denotational definitions try to give meanings to large parts of programs as wholes, making it necessary to regard expressions of the language as denoting functions or state transformations—which in themselves have to be taken as being infinite objects. In order to manipulate infinite objects mathematically, they have to be collected together into spaces or domains of different types (*e.g.* function spaces), and various basic operators have to be defined on these spaces. In order to relate these infinite objects to actual computations, operational considerations are needed to explain how finite approximations to the infinite objects behave under various transformations. The difference here is that the considerations are all abstract and not involved in particular features of the programming language, but again the denotational and operational understandings are given explicit connections that can help to explain both.

In the present work we will examine in some detail the kinds of models which arise in a particular denotational semantic theory as exemplified in, say, [Stoy 1977]—but the models we discuss are more general than those used in the Stoy book. These models are generally called *domains* and the study of such models is called the *theory of domains.* But, of course, that word "domain" is rather bland and does not convey much in itself. We are thus brought to the next question.

What is a domain? Answers to the question abound in the literature. Frequently a domain is taken to be merely a cpo (this notion is defined in the Introduction). In many places more restrictive conditions are imposed. For example, [Scott 1982a] defines a domain to be a consistently complete algebraic cpo (or, more accurately, a poset of elements of information systems), whereas Smyth [1983a] takes a domain to be only an algebraic cpo. In other places a domain is taken to be a strongly algebraic cpo or even a retract of a strongly algebraic cpo. Some writers define domains even less restrictively as posets having various completeness properties. Many people find this proliferation of domains a bit confusing.

Perhaps the source of the problem is the assumption that there is *one* Category of Domains and a goal of domain theory is to find out what it is. If we viewed the use of the term "domain" as we view the use of the term "universe", then we would not be inclined to ask the question, "What is a domain?", without an assumption about the context in which the question is asked. The following definition seems natural: a *domain* is an element of the class of structures (or types) which are being used to give a semantics for a formal language. The choice of the category of domains is therefore determined by the purposes of the semantics. The category will probably need to satisfy certain conditions but it is best to keep it as simple as possible. It is therefore desirable to have a *selection* of possible categories of domains and a theory which describes their properties. With such a selection a proper choice can be made based on a balance between conflicting needs.

What is a domain equation? In giving a semantics for a pragramming language the needed domain of denotations is usually specified via an equation involving the basic operations on semantic domains. These equations can be quite complex and usually involve some form of recursion. For example, the S-expressions of LISP satisfy the equation $S \cong At + (S \times S)$ and solutions to the equation $D \cong At + [D \to D]$ (where $[D \to D]$ denotes the "function space" of D) provide interesting models for the untyped λ-calculus. Generally speaking, a domain equation is defined to be a set of equations of the form:

$$\begin{aligned} D_1 &\cong F_1(D_1, \ldots, D_n) \\ &\vdots \\ D_n &\cong F_n(D_1, \ldots, D_n) \end{aligned}$$

Here the $F_1, \ldots, F_n$ are operators on domains such as

$$F(S) = At + (S \times S) \text{ or } F(D) = At + [D \to D].$$

This list of equations can, in most cases, be reduced to one equation by using the product category, so, for such cases, it is sufficient to restrict attention to the solution of a single equation involving a multi-functor. Getting a solution to such an equation can be problematic, but there is a very good theory which offers general conditions which, when satisfied, assert existence of a canonical solution and *usually* provide a nice, non-trivial solution in an effective way.

1.2 A brief history of the theory of domains

Research in the theory of domains began in Oxford and was continued at Oxford and Princeton in the late 1960's and early 1970's with the work of Christopher Strachey and Dana Scott. Strachey had, for some time before, been attempting to work out a theory of programming-language semantics, but there arose various difficulties with the mathematical foundations of the theory he was deriving. Scott first suggested a replacement for some of the tools which Strachey was using (such as the untyped λ-calculus) by a theory of operators on partial orderings using well-known ideas from recursive function theory. As he explained in [Scott 1977] and in his introduction to Stoy's book [1977], however, the structures (or domains of definition of the operators) were soon found to have great flexibility in definition. Indeed, it proved possible not only to solve recursive domain equations, but also to incorporate the function-space functor into the definitions. In this way the model theory for the untyped λ-calculus was put on solid mathematical ground, and much of Strachey's original approach was surprisingly justified. This was how they succeeded in finding an adequate foundation for a form of programming-language semantics which is now called the "Scott-Strachey" theory. The foundational "Scott part" of the theory is usually refered to as "domain theory" and the "Strachey part" as "denotational semantics." The method as a whole, however, must be considered as joint work.

The groundwork for the theory was done in the winter of 1969, and two early papers on the approach are [Scott 1970] and [Scott and Strachey 1971]. The first published example of a model of the untyped λ-calculus occurs in [Scott 1972]. The latter accomplishment used complete lattices and an inverse limit construction to solve the domain equation

$$D_\infty \cong \mathbf{CPO}(D_\infty, D_\infty)$$

where $\mathbf{CPO}(D, E)$ is the complete lattice of continuous functions between complete lattices D and E. We will say much more about inverse limits and the solution of such equations below.

Another important model was introduced by Gordon Plotkin [1972]. Scott then modified the form of Plotkin's construction slightly to get what he called the "graph model" which he used to derive a very detailed analysis [1976] of the use of ω-continuous lattices (which are the retracts of the graph model) for domain theory. (Actually, the ideas were already known in recursive function theory under the name of *enumeration operators*, but the exact connection with λ-calculus had not been realized before.) Later Plotkin [1978b] introduced a related and somewhat more natural model called T^ω and carried out a similar study for the coherent ω-continuous cpo's. Stoy [1977], however, used Scott's graph model foundation to provide a nice exposition in his book of the use of domain theory in denotational semantics as it now exists. More recently McCracken [1982] used a similar construction to provide a finitary retract model for the polymorphic λ-calculus, which also has importance for programming-language semantics. This quick review hardly touches on the extent of the literature, and the reader must consult the sources mentioned as well as such books as [Barendregt 1984] for a more complete exposition.

Plotkin [1976] introduced a quite different category of domains which he called **SFP**. This is a "large" category and was needed because the more established categories did not have the desired closure properties. Thereafter, it was soon discovered that the property of having least upper bounds is not preserved by the *convex powerdomain* which Plotkin introduced in order to give a denotational semantics for a kind of parallel construction. In [Smyth 1978] a nicer exposition of the convex powerdomain construction is offered and a second functor—the *upper powerdomain*—is introduced.

Much important progress in the development of the applications of categorical notions to domain theory also has taken place. In particular there was a recognition of the importance of the notion of *cartesian closure* for a category of domains. Since there is a correspondence between cartesian closed categories (ccc) and models of the *typed* λ-calculus (see e.g. [Lambek 1980] and [Scott 1980b]), this condition on a category was ideal for the purposes of domain theory. [1] (Scott [1972] already had shown that the continuous lattices formed a ccc.) Moreover, it has been shown that if an object in a ccc has its own function space as a retract then it is a model of the *untyped* λ-calculus! (See [Koymans 1982] for a recent exposition.)

Other unifying categorical themes have been discussed in various places. For example Plotkin and Smyth [1983b] studied the solution of recursive domain equations at a very satisfying level of generality through the use of Wand's concept of an **O**-category (see also [Wand 1979]). The naturality of the choices of categories and functors has been reenforced by results such as Smyth's Theorem [1983] which states that **SFP** is the largest cartesian closed category of ω-algebraic cpo's. Hennessy and Plotkin [1979] have shown that the three powerdomains (upper, lower, and convex) can be characterized as free algebras in certain naturally motivated categories (see [Plotkin 1978a] for details).

A large body of research has also been concentrated on the important task of developing a satisfactory theory of *computability* for domains. Most of Scott's papers discuss this topic in one degree or another, and several papers make this their central objective. Recently, Weihrauch and Deil [1980] proved a generalization of the Myhill-Shepherdson Theorem which applies to a substantive class of continuous cpo's. Winskel and Larsen [1984] discussed the problem of deriving effective solutions to recursive domain equations. Kanda [1980] and Tang [1984] have studied computability for **SFP** and the category of retracts of **SFP** respectively. McCarty [1984] used the Kleene realizability model as an intuitionistic set theory in which

[1] Of course, not just any ccc will do for domain theory. For example, the category of sets forms a very nice ccc. It is unsatisfactory, however, because it is usually impossible to solve recursive domain equations involving the function space functor (owing to obvious cardinality considerations). For instance, no non-trivial set can have its own function space as a retract.

all of the functions are continuous and computable. This approach allows one to show the existence of a functor or function intuitionistically and then (if one desires) to transfer this result to the usual classical set theory. This method produces automatically a computable functor or function, thus eliminating the need to carry out detailed computations with indices for computable functions in order to prove computability. Since the theorem McCarty proved is essentially the same as the Myhill-Shepherdson Theorem, it is likely that these results can be extended to **SFP** and beyond.

1.3 Cartesian closed categories

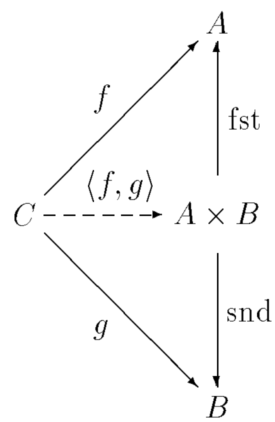
As Scott warns in his paper *Domains for Denotational Semantics*,

> Another word about Category Theory: I actually feel that it is particularly significant for the theory and for the whole area of semantics. But it must be approached with great caution, for the sheer number of definitions *and* axioms can try the most patient reader. It seems to me to be especially necessary in discussing applications of abstract mathematical ideas to keep the motivation strongly in mind. This is hard to do if the categories get too thick but of course it all depends on the writer.

Scott has recently written a spate of papers and monographs [1981a, 1981b, 1982a, and 1982b] which try to bridge the gap between theory and practice, making domain theory more usable by the practicing computer scientist. There is still a lot that needs to be done in this direction, however. We hope that an elegant and well developed theory which has a sensitivity to possible applications will succeed in making domain theory a less abstruce and elite subject. In the present paper we try to restrict the use of category theory to those instances in which it is truely helpful in explaining the basic concepts of domain theory and try to avoid using categorical concepts "because they are there." We assume a low-level familiarity with the notions of category, functor and equivalence of categories and make an effort to define everything else as we go along. If these definitions seem too barren, then the standard references on the subject are [Arbib and Manes 1975], [Herrlich and Strecker 1973] and [MacLane 1971].

We use upper case roman letters like $A, B, \ldots$ to denote objects in a category and lower case roman letters like $f, g, \ldots$ to denote arrows. The notation $f : A \to B$ indicates that f is an arrow with domain A and codomain B. If $\mathbf{C}$ is a category then $\mathbf{C}(A, B)$ is the set of arrows $f : A \to B$. In what follows, $\mathbf{C}(A, B)$ will usually possess additional structure; indeed, almost all of the categories we discuss below are **O**-categories in the sense of [Wand 1979] or [Smyth and Plotkin 1982]. For reasons that we only briefly touched upon above, the following definition is crucial.

A binary operation $\times$ on a category $\mathbf{C}$ is said to be a *product* on $\mathbf{C}$ if for every pair A, B of objects there are arrows fst and snd such that for every pair f, g of arrows there is a unique arrow $\langle f, g \rangle$ which makes the following diagram commute.



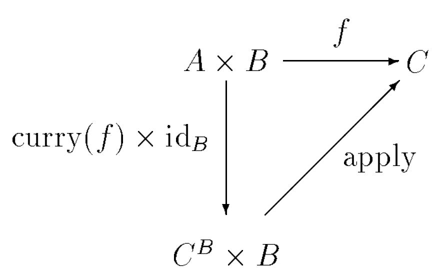
If $f : A \to B$ and $g : A' \to B'$ are arrows in the given hom sets, we define

$$f \times g : A \times A' \to B \times B'$$

by $f \times g = \langle f \circ \mathrm{fst}, g \circ \mathrm{snd} \rangle$. An object 1 in a category $\mathbf{C}$ is *terminal* if for each object A there is a unique arrow $1_A : A \to 1$. A category $\mathbf{C}$ together with a terminal object and product operation is said to be *cartesian*. A cartesian category with a binary operation B^A, called exponentiation is said to be *closed* if for any triple A, B, C of objects there is an arrow $\mathrm{apply} : C^B \times B \to C$ such that for every $f : A \times B \to C$ there is a unique arrow

$$\mathrm{curry}(f) : A \to C^B$$

such that the following diagram commutes.



If a category $\mathbf{C}$ is cartesian and $\mathbf{C}' \subseteq \mathbf{C}$ is a full subcategory such that

1. $A \times B$ is an object in $\mathbf{C}'$ whenever A and B are objects in $\mathbf{C}$ and

2. 1 is an object in $\mathbf{C}'$

then $\mathbf{C}'$ is itself a cartesian category. Moreover, if $\mathbf{C}$ is a cartesian closed category, $\mathbf{C}'$ satisfies 1, 2, and B^A is an object in $\mathbf{C}'$ whenever A, B are objects in $\mathbf{C}'$, then $\mathbf{C}'$ is cartesian closed. Both of these "inheritance properties" follow from the fact that equations that hold in a category will also hold in any full subcategory so the equations that hold for fst, snd, curry and apply in **C** must also hold in $\mathbf{C}'$. We will frequently use this observation below without mentioning it explicitly.

Strictly speaking, a cartesian closed category is more structured than a category because the product, exponential, *etc.* must be *specified.* But generally, we will have in mind a particular product and a particular exponential. Usually these will be determined via the inheritence property mentioned above. That is, if **C** is a ccc and $\mathbf{C}'$ is a full sub-category then we say that $\mathbf{C}'$ is a ccc if it is a ccc with the inherited functors from **C**. So statements like "**C** is cartesian closed" should be taken to mean "With the evident choices of product and exponential, **C** is cartesian closed".

In a cartesian closed category **C**, the operator curry defines a bijection between $\mathbf{C}(A \times B, C)$ and $\mathbf{C}(A, C^B)$ for any objects A, B, C. Hence, in particular, there is a bijection between $\mathbf{C}(1 \times A, B)$ and $\mathbf{C}(1, B^A)$. This supports the intuition that $F(A, B) = B^A$ is the *function space operation* for the category **C**. In the case that $\mathbf{C}(A, B)$ has additional structure, however, care must be taken not to confuse $\mathbf{C}(A, B)$ with the object B^A.

Chapter 2

Representations of Algebraic Cpo's

There are times when it is easier, conceptually, to deal with a representation of a class of spaces rather than with the abstract class itself. For example, thinking about fields of sets is frequently easier than thinking about models of the axioms for boolean algebras. Yet essentially anything one shows about fields of sets also applies to boolean algebras since every field of sets is a boolean algebra and every boolean algebra is isomorphic to a field of sets. Another class which can be studied representationaly is that of algebraic cpo's which we define in this chapter. By using the much more primitive notion of a pre-order we can represent the algebraic cpo's via the ideal completion functor. This allows one to derive properties of the latter class from the representation rather than relying solely on the axioms. Representation has its faults, however. It can make the class of spaces less abstract but it can also make them more difficult to work with by being overly resrictive. It may be difficult, for example, to represent important operations or constructions in a natural way. So a good representation should make the right compromise with abstraction to achieve the most conceptually apealing and flexible result.

Another helpful aspect of a well-described, easy to use class of spaces is that of *intrinsic characterization*. For example, it would be a bit unsatisfying to be told that a group is an algebra which is isomorphic to a certain kind of subalgebra of a permutation group. In a way this is the dual notion to representation; while we are pleased to have the Caley Theorem, thinking of groups as subgroups of permutation groups is not always convenient. It is therefore desirable to describe the class in question in a way that uses as little reference to other classes as possible. We will return to the issue of intrinsic characterization later in the context of categorical descriptions.

2.1 Pre-orders and algebraic cpo's

The fact that algebraic lattices correspond to ideal completions of pre-orders has been known for some time (see, for example, [Birkhoff 1940]); we will extend that correspondence to cases in which least upper bounds may not exist. To get a useful equivalent category one must see what the arrows on the pre-orders should be and find the right functor to carry these arrows over to the continuous functions. The notion of an approximable relation, suggested by Dana Scott, is simple, elegant and meets these conditions quite nicely. The definition of an approximable relation given here generalizes the definitions in the literature to deal with *arbitrary* pre-orders. [1] No more general construction than the one given below seems possible since the full category of algebraic cpo's is characterized in this way.

A *pre-order* is a pair $\langle A, \vdash_A \rangle$ where $\vdash_A$ is a binary relation satisfying the following axioms for each $X, Y, Z \in A$:

1. $X \vdash_A X$;

2. if $X \vdash_A Y$ and $Y \vdash_A Z$ then $X \vdash_A Z$.

It is intended that the "larger" element is the one on the left side of the turnstile. Note that $A = \emptyset$ is allowed. To conserve notation we write $A = \langle A, \vdash_A \rangle$ and when A is clear from context the subscript is dropped. A set $S \subseteq A$ is *bounded* if there is an $X \in A$ such that $X \vdash Y$ for every $Y \in S$. Such an X is called a *bound* for S and we write $X \vdash S$. Trivially, any $X \in A$ is a bound for the empty set. A subset $M \subseteq A$ of a pre-order A is *directed* if every finite subset of M has a bound in M. Note, in particular, that every directed set is non-empty. A subset $M \subseteq A$ is *filtered* if for every finite $u \subseteq M$, there is a $X \in M$ such that $Y \vdash X$ for each $Y \in u$.

Definition: An *approximable relation* $f : A \to B$ is a subset of $A \times B$ which satisfies the following axioms for any $X, X' \in A$ and $Y, Y' \in B$:

1. for every $X \in A$ there is a $Y \in A$ such that $X \ f \ Y$;

2. if $X \ f \ Y$ and $X \ f \ Y'$ then there is a $Z \in B$ such that $X \ f \ Z$ and $Z \vdash_B Y, Y'$;

3. if $X \vdash_A X' \ f \ Y' \vdash_B Y$ then $X \ f \ Y$. $\Box$

Let $g : A \to B$ and $f : B \to C$ be approximable relations. We define a binary relation $f \circ g$ on $A \times C$ as follows. For each $X \in A$ and $Z \in C$, $X \ (f \circ g) \ Z$ if and only if there is a $Y \in B$ such that $X \ g \ Y$ and $Y \ f \ Z$. Also, for each pre-order A define $\mathrm{id}_A = \vdash_A$. It is easy to verify that $f \circ g$ and id_A are approximable relations. With this composition and identity relation the class of pre-orders and approximable relations form a category which we denote by **PO**. We let $\mathbf{PO}(A, B)$ be the set of approximable relations on $A \times B$. The approximable relations are partially ordered by set theoretic inclusion.

For pre-orders A and B we define the *product pre-order*

$$\langle A \times B, \vdash_{A \times B} \rangle$$

as follows:

- $A \times B = \{(X, Y) \mid X \in A \text{ and } Y \in B\}$;

- for any (X, Y), $(X', Y') \in A \times B$, $(X, Y) \vdash_{A \times B} (X', Y')$ if and only if $X \vdash_A X'$ and $Y \vdash_B Y'$.

Suppose A and B are pre-orders. Define relations,

[1] Although exactly this definition and many of the related results do appear in Gordon Plotkin's unpublished Pisa lecture notes [1978a]

- fst : $A \times B \to A$ by (X, Y) fst X' if and only if $X \vdash_A X'$, and
- snd : $A \times B \to B$ by (X, Y) snd Y' if and only if $Y \vdash_B Y'$.

It is easy to check that fst and snd are approximable. Suppose $f : C \to A$ and $g : C \to B$ are approximable relations and define $\langle f, g\rangle : C \to A \times B$ by: $Z\ \langle f, g\rangle\ (X, Y)$ if and only if $Z\ f\ X$ and $Z\ g\ Y$. It is straight-forward to check that $\langle f, g\rangle$ is approximable and $\times$ is a product in the category of pre-orders and approximable relations. If we take 1 to be the single element pre-order, then for each pre-order A there is a unique approximable relation $1_A : A \to 1$. Thus the pre-orders and approximable relations form a cartesian category. Moreover, the empty poset 0, is initial in this category, *i.e.* for any object A there is a unique arrow $0_A : 0 \to A$. This 0_A is the "empty relation" and it is trivially approximable.

A *poset* $\langle D, \sqsubseteq\rangle$ (or *partially ordered set*) is a pre-order that is anti-symmetric, *i.e.* if $x \sqsubseteq y$ and $y \sqsubseteq x$ then $x = y$. Using the established convention we write the "larger" element on the right side of the $\sqsubseteq$ symbol. If $x \sqsubseteq y$ then it is sometimes convenient to write $y \sqsupseteq x$. If $x \sqsubseteq y$ and $x \neq y$ then we write $x \sqsubset y$; we define $\sqsupset$ by a similar convention. A poset $\langle D, \sqsubseteq\rangle$ is said to be a *complete partial order* (cpo) if every directed subset $M \subseteq D$ has a least upper bound $\bigsqcup M \in D$. Let D be a cpo. An element $x \in D$ is *finite* (or compact) if whenever $x \sqsubseteq \bigsqcup M$ for a directed set M, there is a $y \in M$ such that $x \sqsubseteq y$. Let $\mathbf{B}[D]$ denote the set of finite elements of a cpo D. We say that D is *algebraic* if for every $x \in D$, the set $M = \{x_0 \in \mathbf{B}[D] \mid x_0 \sqsubseteq x\}$ is directed and $\bigsqcup M = x$. In other words, in an algebraic cpo every element is the limit of its finite approximations. In a cpo D a set O is said to be *Scott open* if

1. for each $x \in O$, if $y \sqsupseteq x$ then $y \in O$, and
2. if M is a directed set and $\bigsqcup M \in O$ then $M \cap O \neq \emptyset$.

The Scott open sets form a topology on D called the *Scott topology* which we denote by ΣD. Unless we mention otherwise this will always be the assumed topology for a cpo. We define **CPO** to be the category that has cpo's as objects and (Scott) continuous maps as arrows. The full sub-category of algebraic cpo's is denoted **ALG** and the countably based algebraic cpo's by ω**ALG**. In general, when **C** is a subcategory of **ALG** we write ω**C** to mean **C** $\cap\, \omega$**ALG**.

We will frequently wish to transfer a property of pre-orders to a property of posets and conversely. This is usually possible because pre-orders and posets are closely connected. First of all, every pre-order is isomorphic (in the category with approximable relations as arrows) to a poset. To see this, let $\langle A, \vdash\rangle$ be a pre-order. Define an equivalence relation $\sim$ on A by letting $X \sim Y$ if and only if $X \vdash Y$ and $Y \vdash X$. For each X, let $\tilde{X} = \{Y \in A \mid X \sim Y\}$ and set $\tilde{A} = \{\tilde{X} \mid X \in A\}$. If we define a binary relation $\sqsupseteq$ on A by letting $\tilde{X} \sqsupseteq \tilde{Y}$ if and only if $X \vdash Y$, then it is easy to check that $\langle \tilde{A}, \sqsubseteq\rangle$ is a poset and the approximable relation $f : \tilde{A} \to A$ given by $\tilde{X}\ f\ Y$ if and only if $X \vdash Y$ is an isomorphism. In addition, posets are isomorphic in the category with approximable relations as arrows if and only if they are isomorphic in the more familiar category with monotone maps as arrows. We can therefore write $A \cong B$ for pre-orders A and B without fear of ambiguity.

The name "cpo" is inadequate in not saying with respect to what the poset is complete. A more flexible notation used in some places in the literature is to refer to the Γ-*completeness* of poset where Γ is some set of subsets of the poset in question. So the cpo's defined above are *directed* complete posets or dcpo's. This avoids such confusing names as "bounded complete complete posets" but this is the term which is most common in the literature. Since we will not concern ourselves with many of the different sorts of completeness properties we adopt the less flexible, more common notation. The above definition of a cpo does, however, differ from the definitions in the literature in some regards. It does not require that a cpo have a least element; indeed, we do not require a cpo to be non-empty. Much of the usual theory of cpo's goes through for these "bottomless" cases. For example, we may characterize the continous functions on algebraic cpo's in the following way:

Lemma 1 *Let D and E be cpo's. A function $f : D \to E$ is continuous if and only if for every directed set $M \subseteq D$, $f(M)$ is directed and $f(\bigsqcup M) = \bigsqcup f(M)$.* $\square$

The proof of the proposition is well-known and we omit it. See [Barendregt 1981] for further facts about cpo's (with bottoms). Let $\mathbf{CPO}(D, E)$ be the set of continuous functions from D to E. We order $\mathbf{CPO}(D, E)$ by setting $f \sqsubseteq g$ if for every $x \in D$, $f(x) \sqsubseteq g(x)$. It is easy to check that $\mathbf{CPO}(D, E)$ is itself a cpo. By defining an appropriate action on arrows, $\mathbf{CPO}(\cdot, \cdot)$ can be made into a functor on $\mathbf{CPO}$. To see this, suppose $f : D \to E$ and $g : F \to G$ are continuous. Then the function

$$\mathbf{CPO}(f, g) : \mathbf{CPO}(E, F) \to \mathbf{CPO}(D, G)$$

by $\mathbf{CPO}(f, g)(h) = g \circ h \circ f$ is continuous. One can show that if $f' : E \to E'$ and $g' : G \to G'$ then

$$\mathbf{CPO}(f' \circ f, g' \circ g) = \mathbf{CPO}(f, g') \circ \mathbf{CPO}(f', g).$$

Note, in particular, that $\mathbf{CPO}(\cdot, \cdot)$ is contravariant in its first argument. The product of algebraic cpo's is defined exactly as for pre-orders (*i.e.* with the coordinate-wise ordering).

Let $\langle A, \vdash \rangle$ be a pre-order. An *ideal* over A is a directed subset $x \subseteq A$ such that if $X \vdash Y$ and $X \in x$ then $Y \in x$. The *ideal completion* of A is the partial ordering, $\langle |A|, \subseteq \rangle$, of the ideals of A by set-theoretic inclusion. If $X \in A$ then the *principal ideal generated by X* is the set $\downarrow X = \{Y \in A \mid X \vdash Y\}$. Dually, define the *principal filter generated by X* to be the set $\uparrow X = \{Y \in A \mid Y \vdash X\}$. If $S \subseteq A$ then,

$$\downarrow S = \bigcup \{\downarrow X \mid X \in S\}$$
$$\uparrow S = \bigcup \{\uparrow X \mid X \in S\}.$$

Note that for any pre-order A, the set $\{\downarrow X \mid X \in A\}$ of principal ideals over A forms a poset under set inclusion which is isomorphic to A. Moreover, if $A' = \{\uparrow X \mid X \in A\}$ then $\langle A', \subseteq \rangle \cong \langle A, \vdash \rangle$. Note, however, that $X \vdash Y$ if and only if $\uparrow X \subseteq \uparrow Y$ so the ordering on A' is "upside down".

Theorem 2 *If A is a pre-order, then $|A|$ is an algebraic cpo with $\mathbf{B}[|A|] \cong A$. Moreover, every algebraic cpo D is representable in this way because $D \cong |\mathbf{B}[D]|$.*

Proof. Note that if $M \subseteq |A|$ is directed then $\bigcup M$ is the least upper bound of M. Hence $|A|$ is a cpo. If $X \in A$ and $\downarrow X \subseteq \bigcup M$ then $X \in y$ for some $y \in M$ so $\downarrow X \subseteq y$. Hence $\downarrow X$ is finite (as an element of $|A|$). But for any ideal x, the set

$$M = \{\downarrow X \mid X \in x\}$$

is directed (because x is directed) and $x = \bigcup M$. Hence $|A|$ is an algebraic cpo and $\mathbf{B}[|A|] = \{\downarrow X \mid X \in A\}$ is isomorphic to A. On the other hand, if $\langle D, \sqsubseteq \rangle$ is an algebraic cpo then it is easy to verify that $f : D \to |\mathbf{B}[D]|$ by $f(x) = \{x_0 \mid x_0 \sqsubseteq x\}$ is an isomorphism. □

Intuitively, the passage $A \mapsto |A|$ expands A by adding limits for ascending chains. To see this in a specific example, let ${}^{<\omega}2$ be the set of functions $f : n \to 2$ where $n < \omega$. If $f : n \to 2$ and $g : m \to 2$ then say $f \sqsubseteq g$ if and only if $n < m$ and $f(k) = g(k)$ for each $k < n$. the ideal completion $|{}^{<\omega}2|$ of this poset is isomorphic to the union ${}^{<\omega}2 \cup {}^{\omega}2$ where ${}^{\omega}2$ is the set of functions from ω into 2,

- ${}^{<\omega}2$ retains the ordering just mentioned and
- if $f : n \to 2$ and $g : \omega \to 2$ then $f \sqsubseteq g$ if and only if $f(k) = g(k)$ for each $k < n$.

The infinite elements of $|{}^{<\omega}2|$ correspond to those in ${}^{\omega}2$ while the finite elements of $|{}^{<\omega}2|$ correspond to those in ${}^{<\omega}2$. If a poset A has no infinite chains then surely no new elements are added by the ideal completion. We make this intuition precise as follows.

Definition: A poset $\langle A, \sqsubseteq \rangle$ is said to have the *ascending chain condition* (acc) if for every chain $X_0 \sqsubseteq X_1 \sqsubseteq X_2 \sqsubseteq \cdots$ of elements of A there is an $n \in \omega$ such that for every $m \geq n$, $X_m = X_n$. A pre-order $\langle A, \vdash \rangle$ is said to have the acc if $\tilde{A}$ does. □

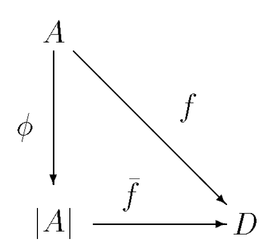
Proposition 3 *If $\langle A, \vdash \rangle$ has the acc then $A \cong |A|$.*

Proof. We show below that $|A| \cong |B|$ if $A \cong B$. Since $A \cong \tilde{A}$ we can therefore assume that A is a poset. We show that each $x \in |A|$ is principal. Assume $x \in |A|$ is *not* principal. Then for each $X \in x$ there is an $X' \in x$ such that $X \sqsubset X'$. But this means there is a chain $X_0 \sqsubset X_1 \sqsubset \cdots$ of elements of x. This contradicts the assumption that A has the acc. Hence $|A| = \{\downarrow X \mid X \in A\} = \mathbf{B}[|A|] \cong A$. □

A rather obvious corollary of the Proposition is that all finite posets are algebraic cpo's. Now, if D is a poset with the acc and $M \subseteq D$ is directed then $\bigsqcup M = x$ for some $x \in M$. Hence, if $f : D \to E$ is monotone then $f(\bigsqcup M) = f(x) = \bigsqcup f(M)$. We conclude that when D has the acc then $\mathbf{CPO}(D, E)$ is just the set of monotone functions from D into E.

There is a sense in which $|A|$ is freely generated by A. Formally, we have the following:

Theorem 4 *Let A be a pre-order and suppose $\phi : A \to |A|$ by $\phi : X \mapsto \downarrow X$. Then for every cpo D and monotone function f there is a unique continuous function $\bar{f}$ such that the following diagram commutes.*



Moreover, the correspondence $f \mapsto \bar{f}$ *is monotone.*

Proof. Let f and D be given as in the theorem. Define $\bar{f}$ by

$$\bar{f}(x) = \bigsqcup\{f(X) \mid X \in x\}.$$

This makes sense because f is monotone, x is directed and D is complete. To see that $\bar{f}$ is continuous, suppose M is a directed subset of $|A|$. Then

$$\begin{aligned}\bar{f}(\bigcup M) &= \bigsqcup\{f(X) \mid X \in \bigcup M\}\\ &= \bigsqcup\{f(X) \mid X \in x \text{ for some } x \in M\}\\ &= \bigsqcup\{\bigsqcup\{f(X) \mid X \in x\} \mid x \in M\}\\ &= \bigsqcup \bar{f}(M).\end{aligned}$$

To see that $\bar{f}$ is unique, suppose $g : |A| \to D$ is continuous and for every $X \in A$, $g(\downarrow X) = f(X)$. Then

$$\begin{aligned}g(x) &= \bigsqcup\{g(\downarrow X) \mid X \in x\}\\ &= \bigsqcup\{f(X) \mid X \in x\}\\ &= \bar{f}(x).\end{aligned}$$

Now, if $f_0 \sqsubseteq f_1$ for monotone functions $f_0, f_1 : A \to D$ then for each $x \in |A|$,

$$\bar{f}_0(x) = \bigsqcup\{f_0(X) \mid X \in x\} \sqsubseteq \bigsqcup\{f_1(X) \mid X \in x\} = \bar{f}_1(x).$$

Hence $\bar{f}_0 \sqsubseteq \bar{f}_1$ and the correspondence $f \mapsto \bar{f}$ is monotone. □

Definition: If A and B are pre-orders and $f : A \to B$ is an approximable relation then define a function $|f| : |A| \to |B|$ by

$$|f|(x) = \{Y \mid X\ f\ Y \text{ for some } X \in x\}.$$

□

Note that the conditions set down in the definition of an approximable relation insure that the set on the right *is* an ideal.

Proposition 5 *Let* A *and* B *be pre-orders. If* $f : A \to B$ *is approximable then* $|f| : |A| \to |B|$ *is continuous. Moreover, the correspondence* $f \mapsto |f|$ *is an isomorphism between the posets* $\mathbf{PO}(A, B)$ *and* $\mathbf{CPO}(|A|, |B|)$.

Proof. To see that $|f|$ is continuous, suppose $M \subseteq |A|$ is directed. Then

$$\begin{aligned}\bigcup |f|(M) &= \bigcup\{|f|(x) \mid x \in M\}\\ &= \bigcup\{\{Y \mid X\ f\ Y \text{ for some } X \in x\} \mid x \in M\}\\ &= \{Y \mid X\ f\ Y \text{ for some } X \in \bigcup M\}\\ &= |f|(\bigcup M)\end{aligned}$$

so $|f|$ is continuous by Lemma 1. Now, suppose $f : |A| \to |B|$ is continuous. Define a relation $f' \subseteq A \times B$ by letting $X\ f'\ Y$ if and only if $Y \in f(\downarrow X)$. For any $x \in |A|$ we have

$$\begin{aligned} |f'|(x) &= \{Y \mid X\ f'\ Y \text{ for some } X \in x\} \\ &= \{Y \mid Y \in f(\downarrow X) \text{ for some } X \in x\} \\ &= \bigcup\{f(\downarrow X) \mid X \in x\} \\ &= f(x) \end{aligned}$$

since f is continuous. On the other hand, if $f \subseteq A \times B$ is approximable then $X\ |f|'\ Y$ if and only if $Y \in |f|(\downarrow X)$ if and only if $X\ f\ Y$. Hence $|f|' = f$. Now, if $f \subseteq g$ for approximable relations f and g then

$$\begin{aligned} |f|(x) &= \{Y \mid X\ f\ Y \text{ for some } X \in x\} \\ &\subseteq \{Y \mid X\ g\ Y \text{ for some } X \in x\}\ . \\ &= |g|(x) \end{aligned}$$

On the other hand, suppose $f, g : |B| \to |A|$ are continuous. If $f \sqsubseteq g$ and $X\ f'\ Y$ then $Y \in f(\downarrow X) \subseteq g(\downarrow X)$ so $X\ g'\ Y$. Hence $f' \subseteq g'$. We conclude that $\mathbf{PO}(A, B) \cong \mathbf{CPO}(|A|, |B|)$. □

Suppose that $g : A \to B$ and $f : B \to C$ are approximable relations. Then for any $x \in |A|$,

$$\begin{aligned} (|f| \circ |g|)(x) &= \{Z \mid Y\ f\ Z \text{ for some } Y \in |g|(x)\} \\ &= \{Z \mid X\ g\ Z \text{ and } Y\ f\ Z \text{ for some } X \in x \text{ and } Y \in B\} \\ &= \{Z \mid X\ (f \circ g)\ Z \text{ for some } X \in x\} \\ &= |f \circ g|(x). \end{aligned}$$

Since $|\mathrm{id}_A|(x) = x$ for any pre-order A and $x \in |A|$ we may conclude that the passage $A \mapsto |A|$, $f \mapsto |f|$ is a *functor*. In category theoretic terminology, Proposition 2 says that this functor is *dense* and Proposition 5 says that it is *full* and *faithful*. We have therefore proved the following:

Proposition 6 *The category of pre-orders and approximable relations is equivalent (in the category theoretic sense) to the category of algebraic cpo's.* □

This equivalence extends to subcategories as well. We make the following:

Definition: If $\mathcal{K}$ is a class of pre-orders then the category $\mathbf{Id}_{\mathcal{K}}$ of ideal completions induced by $\mathcal{K}$ has as objects algebraic cpo's D such that $\mathbf{B}[D]$ is isomorphic to a pre-order in $\mathcal{K}$ and has as arrows continuous functions. For a category of pre-orders $\mathbf{C}$, $\mathbf{Id}_{\mathbf{C}}$ is just $\mathbf{Id}_{\mathcal{K}}$ where $\mathcal{K}$ is the class of objects of $\mathbf{C}$. □

Before concluding this section we comment on one other noteworty equivalence of categories. We begin with the following

Theorem 7 *Let A and B be posets. There is an order isomorphism between monotone maps from A to B and continuous maps $f : |A| \to |B|$ that send finite elements of $|A|$ to finite elements of $|B|$.*

Proof. Let $\phi : A \to |A|$ and $\psi : B \to |B|$ be the principal ideal maps. If $m : A \to B$ is monotone then $\psi \circ m : A \to |B|$ is monotone so by Theorem 4 there is a unique continuous function $\sigma(m) : |A| \to |B|$ such that $\sigma(m) \circ \phi = \psi \circ m$. Apparently $\sigma(m)$ sends principal ideals to principal ideals. On the other hand, if $c : |A| \to |B|$ is continuous and sends principal ideals to principal ideals, then we can define a function $\tau(c) : A \to B$ by setting $\tau(c)(X) = Y$ if and only if $c(\downarrow X) = \downarrow Y$. Now, $\tau(c)$ is monotone and by definition $c \circ \phi = \psi \circ \tau(c)$. Hence, by the uniqueness condition on σ, $c = \sigma \circ \tau(c)$. If $m : A \to B$ is monotone then for every $X \in A$, $\sigma(m)(\downarrow X) = \downarrow m(X)$. So by definition, $[(\tau \circ \sigma)(m)](X) = m(X)$ and therefore $(\tau \circ \sigma)(m) = m$. That σ is monotone follows from Theorem 4. Monotonicity of τ is immediate from its definition. □

Suppose A, B are pre-orders and $f : A \to B$ is monotone (*i.e.* $f(X) \vdash_B f(Y)$ whenever $X \vdash_A Y$). If $\sigma(f) : |A| \to |B|$ is defined as it was in the proof of Theorem 7 then $\sigma(f) = |f^\dagger|$ where $f^\dagger$ is an approximable relation defined by setting $X\ f^\dagger\ Y$ if and only if $f(X) \vdash Y$. By a slight abuse of notation we define $|f|$ to mean $|f^\dagger|$. In particular, for a monotone $f : A \to B$,

$$|f|(x) = \{Y \mid f(X) \vdash Y \text{ for some } X \in x\}$$

where $x \in |A|$. We claim that $f \mapsto f^\dagger$ is a functor (where $A^\dagger = |A|$ is the action on objects). If id_A is the identity *function* on A then $X\ \mathrm{id_A}^\dagger\ Y$ if and only if $X = \mathrm{id}_A(X) \vdash_A Y$. Hence $\mathrm{id}_A^\dagger$ is the (approximable) identity *relation*. Suppose $f : A \to B$ and $g : B \to C$ are monotone functions. If $X\ (f^\dagger \circ g^\dagger)\ Z$ then $X\ g^\dagger\ Y$ and $Y\ f^\dagger\ Z$ for some Y so $g(X) \vdash Y$ and $f(Y) \vdash Z$. By the monotoniity of f, $(f \circ g)(X) \vdash f(Y) \vdash Z$ so $X\ (f \circ g)^\dagger\ Z$. On the other hand, if $X\ (f \circ g)^\dagger\ Z$ then $(f \circ g)(X) \vdash Z$ so $X\ g^\dagger\ (g(X))$ and $(g(X))\ f^\dagger\ Z$. Thus $X\ (f^\dagger \circ g^\dagger)\ Z$. This shows that $f^\dagger \circ g^\dagger = (f \circ g)^\dagger$ and $(\cdot)^\dagger$ is therefore a functor. Since $|\cdot|$ is a functor on approximable relations we know therefore that our definition of $|\cdot|$ on montone functions is also a functor. The proof of the following is therefore quite straight-forward:

Theorem 8 *The category of pre-orders and monotone functions is equivalent to the category of algebraic cpo's and continuous functions which map finite elemtents to finite elements.* □

2.2 Plotkin orders

We now introduce a cartesian closed category of pre-orders called the Plotkin orders. A closely related category called **SFP** was introduced by Gordon Plotkin [1976]. However, Plotkin's orginal presentation is somewhat non-elementary in the sense that it requires an understanding of the inverse limit construction. By working with the Plotkin orders we hope to avoid this level of abstraction while retaining all essential properties. This is done by working with an easy-to-understand "upper bounds" condition on a pre-order and getting the **SFP** objects as ideal completions of pre-orders that satisfy this condition. This idea has been used for other classes as well. In particular, it is exploited extensively by Scott [1981a, 1981b, 1982a] for the consistently complete algebraic cpo's. For these spaces the upper bounds condition is simply consistant completeness. For **SFP**, the use of the appropriate condition allows Smyth [1983] to prove many significant results without ever mentioning the inverse limit construction.

Definition: Suppose A is a pre-order and $S \subseteq A$. We say that S is *normal* in A and write $S \triangleleft A$ if for every $X \in A$ the set $S \cap \downarrow X$ is directed. A is a *Plotkin order* if for every finite $u \subseteq A$, there is a finite $B \supseteq u$ such that $B \triangleleft A$. The category of Plotkin orders with approximable relations will be denoted by **PLT**.

There is a similar condition on pre-orders that is is frequently useful. A set u' of upper bounds of u is said to be *complete* if whenever $X \vdash u$, there is an $X' \in u'$ such that $X \vdash X'$. An upper bound $X \vdash u$ of u is *minimal* if for each Y, $X \vdash Y \vdash u$ implies $X \sim Y$. If every finite subset of A has a complete set of minimal upper bounds then we say that A has the *(weak) minimal upper bounds property* (or "property m"). If every finite subset of A has a finite complete set of minimal upper bounds then we say that A has the *strong* minimal upper bounds property (or "property M"). □

Intuitively, if $S \triangleleft A$ then S offers a directed approximation to every element of A. Thus one might think of S as itself an approximation to A. A pre-order A is a Plotkin order just in case it can be built up as a union of finite approximations. Note, incidently, that if $S \triangleleft A$ and $X \in A$ then $X \vdash \emptyset$ and $\emptyset \subseteq S$, so there is an $X' \in S$ such that $X \vdash X'$. Obviously finite pre-order is a Plotkin order. Indeed, any pre-order having property M and the acc is a Plotkin order. A proof of this latter fact uses König's lemma and can be found in [Smyth 1983]. Many more examples of Plotkin orders will be given in later remarks. As an example of how property m arises, we show that if D is a cpo then D^{op} has property m. For suppose $S \subseteq D$ and $x \sqsubseteq S$. Let L be a maximal chain in $\bigcap\{\downarrow y \mid y \in S\} \cap \uparrow x$ and suppose $x' = \bigsqcup L$. Then x' is a maximal lower bound for S. Hence every element of $\bigcap\{\downarrow y \mid y \in S\}$ lies below a maximal lower bound of S. But this just says that in D^{op}, S has a complete set of minimal upper bounds. Hence D^{op} has property m. Actually, since we did not assume that S is finite, D^{op} satisfies a condition slightly stronger than m, namely: *every* subset of D^{op} has a complete set of minimal upper bounds.

We summarize some of the properties of the $\triangleleft$ relation in the following

Lemma 9 *Let A, B, C be pre-orders.*

1. *Suppose $A \subseteq B$. Then $A \triangleleft B$ if and only if for every $u \subseteq A$ there is a set $u' \subseteq A$ of upper bounds for u which is complete for u in B.*

2. *If $A \triangleleft B \triangleleft C$ then $A \triangleleft C$.*

3. *If $A \subseteq B \subseteq C$ and $A \triangleleft C$ then $A \triangleleft B$.*

Proof. These follow immediately from the definitions. □

Let A be a poset and suppose $u \subseteq A$ is finite. If a complete set u' of upper bounds of u is finite then it contains a complete set of minimal upper bounds. If A is a Plotkin order then there is a finite $B \triangleleft A$ with $u \subseteq B$. Hence, by 9, u has a finite set of minimal upper bounds in A. It follows, therefore, that a Plotkin order has property M. It is not true, however, that every pre-order having property M is a Plotkin order. A counter-example is illustrated in Figure 2.1a. Figures 2.1b and 2.1c illustrate two other ways in which a poset can fail to be a Plotkin order (by failing to have property M).

It is often easier to work with Plotkin orders which are *posets* because in a poset with property M, the set of minimal upper bounds of a finite bounded set is finite and complete.

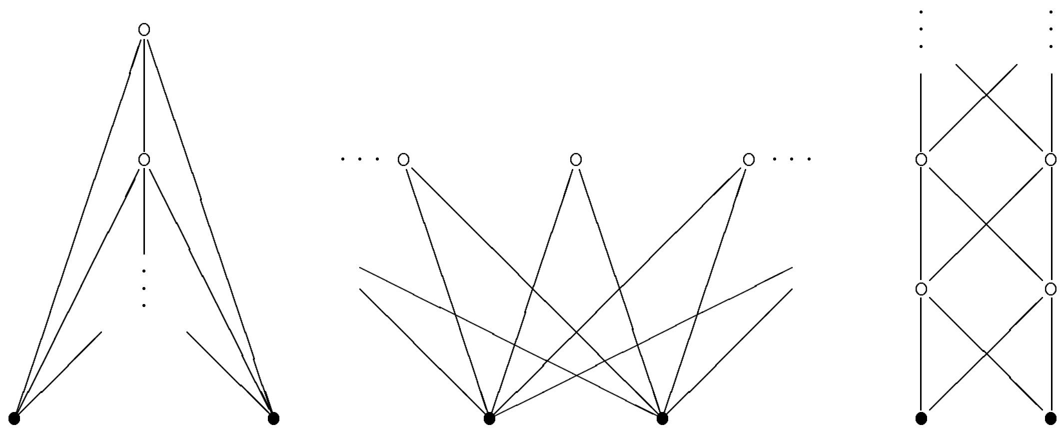


Figure 2.1: Posets that are not Plotkin orders.

Little is lost by this restriction, since every pre-order is isomorphic (in the category with approximable relations as arrows) to a poset $\tilde{A}$ and it is possible to show that A is a Potkin order if and only if $\tilde{A}$ is a Plotkin poset. We might have taken the Plotkin posets as our fundamental notion but this would complicate the definitions of some of the functors, and in any event would narrow the scope of discussion unnecessarily. We will, however, frequently restrict our attention to posets in order to simplify the discussion.

Suppose A is a pre-order. For each $u \subseteq A$, let

$$\mathrm{MUB}_A(u) = \{X \in A \mid X \text{ is a minimal upper bound of } u\}.$$

For each $S \subseteq A$, we define subsets $\mathcal{U}^n_A(S) \subseteq A$, $n \in \omega$, as follows:

$$\begin{aligned} &\mathcal{U}^0_A(S) = S, \\ &\mathcal{U}^{n+1}_A(S) = \{X \mid X \in \mathrm{MUB}_A(u) \text{ for some finite } u \subseteq \mathcal{U}^n_A(S)\}, \\ &\mathcal{U}^*_A(S) = \bigcup_{n\in\omega} \mathcal{U}^n_A(S). \end{aligned}$$

As usual, when A is understood from context we drop the subscripts.

Lemma 10 *If A is a poset with property* m *and $S \subseteq A$, then*

$$\mathcal{U}^*(S) = \bigcap\{B \mid S \subseteq B \lhd A\} \lhd A.$$

Thus, A is a Plotkin poset if and only if A has property m *and for every finite $u \subseteq A$, $\mathcal{U}^*(u)$ is finite.*

Proof. Suppose $S \subseteq B \lhd A$. Then clearly $S = \mathcal{U}^0(S) \subseteq B$. So suppose $\mathcal{U}^n(S) \subseteq B$ and $X \in \mathrm{MUB}(u)$ for some finite $u \subseteq \mathcal{U}^n(S)$. Since $B \lhd A$, there is a $Y \in B$ such that $X \sqsupseteq Y$ and $Y \sqsupseteq u$. But this means $Y = X$ so $X \in B$. Hence $\mathcal{U}^{n+1}(S) \subseteq B$ and we conclude that $\mathcal{U}^*(S) \subseteq B$. To see that $\mathcal{U}^*(S) \lhd A$, let $u \subseteq \mathcal{U}^*(S)$ be finite. Then $u \subseteq \mathcal{U}^n(S)$ for some n. So, if $X \sqsupseteq u$ then $X \sqsupseteq Y$ for some $Y \in \mathrm{MUB}(u) \subseteq \mathcal{U}^{n+1}(X) \subseteq \mathcal{U}^*(S)$. □

Corollary 11 *Let A and B be pre-orders with $B \lhd A$. If A has property* m (M) *then B has property* m (M). *Moreover, if A is a Plotkin order then so is B.*

Proof. Suppose A has property m and $B \lhd A$. If $u \subseteq B$ is finite then the minimal upper bounds of $u \subseteq A$ must lie in B by Lemma 10. Since those upper bounds form a complete set for u in A, they form a complete set for u in B. Since u was arbitrary, it follows that B has property m. The proof for property M is essentially the same. Suppose A is a Plotkin order, $B \lhd A$ and $u \subseteq B$ is finite. Since A has property m (by the lemma), B must also have property m. But then $\mathcal{U}^*_A(u) \subseteq B$ so $\mathcal{U}^*_B(u) = \mathcal{U}^*_A(u)$ is finite. Hence B is a Plotkin order. □
Definition: If a pre-order A has property m then we define the *root*, of A to be $\mathcal{U}^*_A(\emptyset)$. □

2.3 The exponential on PLT

Definition: Let A and B be pre-orders. We define the *exponential pre-order*

$$\langle B^A, \vdash_{B^A} \rangle$$

as follows:

- $p \in B^A$ if and only if p is a finite non-empty subset of $A \times B$ such that for every $Z \in A$, the set
$$\{(X, Y) \in p \mid Z \vdash_A X\}$$
has a maximum with respect to the ordering on $A \times B$.
- $p \vdash_{B^A} q$ if and only if for every $(X, Y) \in q$ there is a pair $(X', Y') \in p$ such that $X \vdash_A X'$ and $Y' \vdash_B Y$. □

The intuition behind the exponential is that each $p \in B^A$ is a finite piece of an approximable relation. The complexity of the first part of the definition is due to the fact that p must be "complete" enough to fully specify what is happening at the minimal upper bounds of finite subsets of its domain. As a consequence of this "completeness", we can show that there is a correspondence between equivalence classes of elements $p \in B^A$ and approximable relations that are finite in the subset ordering on $\mathbf{PO}(A, B)$. Note that if $p \in B^A$ then

$$\{X \mid (X, Y) \in p\} \lhd A.$$

Perhaps it is more intuitive to understand the elements of B^A in terms of the familiar concept of a *step function.* If $p \in B^A$, define $\mathrm{step}_p : \tilde{A} \to \tilde{B}$ by

$$\mathrm{step}_p(\tilde{Z}) = \max\{\tilde{Y} \mid Z \vdash X \text{ and } (X, Y) \in p\}.$$

Then step_p is a monotone function and for each $p, q \in B^A$, $\mathrm{step}_p \sqsupseteq \mathrm{step}_q$ if and only if $p \vdash_{B^A} q$.

Lemma 12 *If $f : A \to B$ is approximable and $M \lhd A$, $N \lhd B$ are finite then $f \cap (M \times N)$ is an element of B^A.*

Proof. Let $X \in A$. Since $M \triangleleft A$ there is an $X_0 \in M$ such that $X \vdash_A X_0 \vdash_A M \cap \downarrow X$. If $v = \{Y \in N \mid X_0\ f\ Y\}$ then because f is approximable, there is a $Y \in B$ such that $Y \vdash_B v$ and $X_0\ f\ Y$. Since $N \triangleleft B$ there is a $Y_0 \in N$ such that $Y \vdash_B Y_0 \vdash_B N \cap \downarrow Y$ Since f is approximable we know also that $X_0\ f\ Y_0$. The conditions of 1. in the definition are therefore satisfied. □

Proposition 13 *Let A and B be pre-orders. Then*

1. *If $M \triangleleft A$ and $N \triangleleft B$ are finite then $N^M \triangleleft B^A$.*

2. *If A and B are Plotkin orders, then B^A is a Plotkin order.*

Proof. 1. Let $p \in B^A$ and set $q = \{(X,Y) \in M \times N \mid X\ f_p\ Y\}$ where

$$f_p = \{(X',Y') \in A \times B \mid X' \vdash_A X \text{ and } Y \vdash_B Y' \text{ for some } (X,Y) \in p\}.$$

We check the three conditions for approximablility of f_p. First, if $X \in A$ then there is an $(X',Y') \in p$ such that $X \vdash_A X'$. Hence $X\ f_p\ Y'$ For the second condition, suppose $X\ f_p\ Y_0$ and $X\ f_p\ Y_1$. Let $(X_0',Y_0'),(X_1',Y_1') \in p$ be such that $X \vdash_A X_0', X_1'$ and $Y_0' \vdash_B Y_0$ and $Y_1' \vdash_B Y_1$. Since $p \in B^A$ there is a pair $(X',Y') \in p$ such that $X \vdash_A X'$ and $X' \vdash_A X_0', X_1'$ and $Y' \vdash_B Y_0', Y_1'$. Hence $X\ f_p\ Y'$ and $Y' \vdash_B Y_0, Y_1$. To get the third condition, note that if $X \vdash_A X'$ and $X'\ f_p\ Y'$ and $Y' \vdash_B Y$ then $X\ f_p\ Y$ follows immediately from the definition of f_p. Since f_p is approximable, $q \in B^A$ by Lemma 12. It follows directly from the definition of q that $p \vdash_{B^A} q$. If $p \vdash_{B^A} r$ and $r \in N^M$ then $r \subseteq q$ so $q \vdash_{B^A} r$. Hence $N^M \triangleleft B^A$.

2. Suppose u is a finite subset of B^A. Since A and B are Plotkin orders, there are finite subsets $M \triangleleft A$ and $N \triangleleft B$ such that

$$\{X \mid (X,Y) \in u \text{ for some } Y \in B\} \subseteq M, \text{ and}$$
$$\{Y \mid (X,Y) \in u \text{ for some } X \in A\} \subseteq N.$$

By 1., $N^M \triangleleft B^A$. Since $u \subseteq N^M$ and N^M is finite the result follows. □

Proposition 14 *Let A and B be pre-orders. Then*

1. *If $M \triangleleft A$ and $N \triangleleft B$ then $M \times N \triangleleft A \times B$.*

2. *If A and B are Plotkin orders then $A \times B$ is a Plotkin order.*

Proof. 1. Suppose $u \subseteq M \times N$ is finite and $(X,Y) \vdash u$. Say

$$\mathrm{fst}(u) = \{X' \in A \mid (X',Y') \in u \text{ for some } Y' \in B\}, \text{ and}$$
$$\mathrm{snd}(u) = \{Y' \in B \mid (X',Y') \in u \text{ for some } X' \in A\}.$$

Then $X \vdash \mathrm{fst}(u)$ and $Y \vdash \mathrm{snd}(u)$ so there are $X' \in M$ and $Y' \in N$ such that $X \vdash X' \vdash \mathrm{fst}(u)$ and $Y \vdash Y' \vdash \mathrm{snd}(u)$. Hence $(X,Y) \vdash (X',Y') \vdash u$ and $(X',Y') \in M \times N$.

2. Similar to the proof of part 2 of 13. □

Since the single element pre-order 1 is a Plotkin order, by Lemma 13 and Lemma 14, **PLT** is a cartesian category. For pre-orders B and C, define a relation apply $\subseteq (C^B \times B) \times C$ by

$$(p,X) \text{ apply } Y \text{ iff } \exists (X',Y') \in p.\ X \vdash_A X' \text{ and } Y' \vdash_B Y.$$

We now check the conditions for approximability of apply.

1. Since p is non-empty for any $p \in C^B$ and (p, X) apply Y for any $(X, Y) \in p$, we know that for any $(p, X) \in C^B$ there is a $Y \in C$ such that (p, X) apply Y.

2. Suppose (p, X) apply Y_0 and (p, X) apply Y_1. Say $(X_0', Y_0'), (X_1', Y_1') \in p$ such that $Y_0' \vdash_C Y_0$, $Y_1' \vdash_C Y_1$, and $X \vdash_B X_0', X_1'$. Since $p \in C^B$, there is an $(X', Y') \in p$ such that $X \vdash_B X'$, $X' \vdash_B X_0', X_1'$ and $Y' \vdash_C Y_0', Y_1'$. By the definition of apply we can conclude that (p, X') apply Y'. Hence, (p, X) apply Y'.

3. Now, suppose $(p, X) \vdash_{C^B \times B} (p_0, X_0)$, (p_0, X_0) apply Y_0, and $Y_0 \vdash_C Y$. To show that (p, X) apply Y, we must find $(X', Y') \in p$ such that $X \vdash_B X'$ and $Y' \vdash_C Y$. By the definition of apply, there is a pair $(X_0', Y_0') \in p_0$ such that $X_0 \vdash_B X_0'$ and $Y_0' \vdash_C Y_0$. Since $p \vdash_{C^B} p_0$, there is a pair $(X', Y') \in p$ such that $X_0' \vdash_B X'$ and $Y' \vdash_C Y_0'$. By the transitivity of $\vdash_B$ and $\vdash_C$, this is the pair we are looking for. We may conclude that apply is approximable.

Theorem 15 *If $f : A \times B \to C$ is approximable and A, B and C be Plotkin orders then there is a unique approximable relation* curry$(f) : A \to C^B$ *such that*

$$\text{apply} \circ (\text{curry}(f) \times \text{id}_B) = f.$$

Hence **PLT** *is cartesian closed.*

Proof. Suppose $X \in A$ and $p \in C^B$. Define curry(f) by

$$X \ \text{curry}(f)\ p \text{ iff } \forall (Y, Z) \in p.\ (X, Y)\ f\ Z.$$

We must show that curry(f) is approximable. Note that the relation $g \subseteq B \times C$, given by $Y\ g\ Z$ iff $(X, Y)\ f\ Z$, is approximable.

1. Let $X \in A$ and suppose $B' \lhd B$, $C' \lhd C$ are finite. Then by Lemma 12, $p = g \cap (B' \times C') \in B^C$ so by the definition of g, X curry(f) p.

2. Let $p_0, p_1 \in C^B$ and suppose X curry(f) p_0 and X curry(f) p_1 for some $X \in A$. Since B and C are Plotkin orders, it is possible to find pre-orders $B' \lhd B$ and $C' \lhd C$ such that $p_0 \cup p_1 \subseteq B' \times C'$. By Lemma 12, $p = g \cap (B' \times C') \in B^C$. Of course, $p \vdash_{C^B} p_0, p_1$ and X curry(f) p.

3. Now suppose X' curry(f) p', $X \vdash_A X'$, and $p' \vdash_{C^B} p$. If $(Y, Z) \in p$ then there is a pair $(Y', Z') \in p'$ such that $Y \vdash_B Y'$ and $Z' \vdash_C Z$. Since X' curry(f) p' we have $(X', Y')\ f\ Z'$. But $(X, Y) \vdash_{A \times B} (X', Y')$ and $Z' \vdash_C Z$ so $(X, Y)\ f\ Z$. This shows that X curry(f) p. So we may conclude that curry(f) is approximable.

To see that apply $\circ$ (curry$(f) \times \text{id}_B) = f$, take $(X, Y) \in A \times B$ and $Z \in C$ such that $(X, Y)\ f\ Z$. Since C and B are Plotkin orders, by Lemma 12 there is a $p \in C^B$ with $(Y, Z) \in p \subseteq f$. Thus X curry(f) p and (p, Y) apply Z, so

$$(X, Y) \ \text{apply} \circ (\text{curry}(f) \times \text{id}_B)\ Z. \tag{$*$}$$

On the other hand, suppose equation $(*)$ holds. Then there is a $p \in C^B$ such that X curry(f) p and (p, Y) apply Z. By the definition of apply, there is a pair $(Y', Z') \in p$ such that $Y \vdash_B Y'$ and (X, Y') f Z'. Now, X curry(f) p implies (X, Y') f Z'. Hence (X, Y) f Z.

To show that curry(f) is unique, let $g : A \to C^B$ and $h : A \to C^B$ be approximable relations such that

$$\text{apply} \circ (g \times \text{id}_B) = \text{apply} \circ (h \times \text{id}_B).$$

If X g p and $(Y, Z) \in p$ then (X, Y) apply $\circ\, (g \times \text{id}_B)$ Z so (X, Y) apply $\circ\, (h \times \text{id}_B)$ Z. Thus X h q for some $q \in C^B$ with a pair $(Y', Z') \in q$ such that $Y \vdash_B Y'$ and $Z' \vdash_C Z$. If we generate such a q for each $(Y, Z) \in p$ then we can use them, together with the fact that h is approximable, to show that there is an $r \in C^B$ such that X h r and $r \vdash_{C^B} p$. Hence $g \subseteq h$. A similar argument will show that $h \subseteq g$. □

Corollary 16 *If A and B are Plotkin orders, then $|B^A| \cong \mathbf{CPO}(|A|, |B|)$.*

Proof. By Proposition 5 we know that $\mathbf{PO}(A, B) \cong \mathbf{CPO}(|A|, |B|)$. It is also clear that

$$\mathbf{PO}(1 \times A, B) \cong \mathbf{PO}(A, B), \text{ and}$$
$$\mathbf{PO}(1, B^A) \cong |B^A|.$$

By Theorem 15 we know that curry defines a bijection

$$\text{curry} : \mathbf{PO}(1 \times A, B) \to \mathbf{PO}(1, B^A)$$

(with inverse $g \mapsto \text{apply} \circ (g \times \text{id})$). The fact that curry and its inverse are monotone follows immediately from their definitions. □

If F is a functor on a category $\mathbf{C}$ of pre-orders then F induces a functor $|F|$ on $\mathbf{Id_C}$ as follows. For an object $|A|$ in $\mathbf{Id_C}$, $|F|(|A|) = |F(A)|$ and if $f : A \to B$ is approximable then we define $|F|(f) = |F(f)|$. A similar set of definitions applies to multiary functors such as the exponential and product. In particular, Corollary 16 shows that if $F(\cdot, \cdot)$ is the exponential functor then $|F|$ is the functor $\mathbf{CPO}(\cdot, \cdot)$. A simple argument will also show that $|A \times B| \cong |A| \times |B|$ so the product functions on $\mathbf{C}$ and $\mathbf{Id_C}$ are the usual ones. It is one of our primary themes to demonstrate that this equivalence between the functor categories can be helpful in studying the properties of functors defined on various classes of algebraic cpo's. The equivalence evidently shows that $\mathbf{Id_{PLT}}$ is a cartesian closed category. We will use the equivalence frequently below to study other functors and classes.

Chapter 3

The Category of Profinite Domains

There are many instances in which it is helpful to look at functions between cpo's which satisfy conditions stronger than continuity. In particular, the class of continuous *projections* and the corresponding class of *embeddings* play a crucial role in the solution of recursive domain equations. These important classes of continuous functions suggest in a natural way the choice of a particular category of cpo's which we call *profinite domains*. The first section of the chapter discusses projections and embeddings and how they are generalized by the notion of an adjunction. We also present an interesting and important functor which associates with an algebraic cpo its cpo of algebraic deflations.

For a category to serve as an appropriate universe of semantic domains there are two primary conditions it must satisfy. First, it must be closed under the basic operations which are being used to build types, and second it must have (canonical) solutions for equations being used to specify denotations. Both of these considerations will be discussed in later chapters. In the second section of the current chapterq we lay the goundwork for such a discussion. To this end we present a simple set of conditions whereby the closure under certain limits of a class of cpo's can be determined. By using the ideal completion correspondence discussed in Chapter 2, the condition can be made especially simple if the cpo's in the class are algebraic.

3.1 Adjunctions between posets

Let A and B be posets and suppose $p : B \to A$ and $q : A \to B$ are monotone maps. If $p \circ q \sqsupseteq \mathrm{id}_A$ and $q \circ p \sqsubseteq \mathrm{id}_B$ then p is said to be an *upper adjoint* and q a *lower adjoint*. The pair $\langle p, q \rangle$ is said to be an *adjunction* (or *Galois connection*) from B to A. If $\langle p, q \rangle$ is an adjunction and $p \circ q = \mathrm{id}_B$ then p is said to be a *projection*, q an *embedding*, and $\langle p, q \rangle$ a *pe-pair*. If, on the other hand, $\langle p, q \rangle$ is an adjunction such that $q \circ p = \mathrm{id}_A$ then $\langle p, q \rangle$ is said to be a *closure*. We write $\langle p, q \rangle : B \xrightarrow{\mathrm{adj}} A$, $\langle p, q \rangle : B \xrightarrow{\mathrm{pe}} A$, or $\langle p, q \rangle : B \xrightarrow{\mathrm{clo}} A$ to indicate that $\langle p, q \rangle$ is an adjunction, pe-pair or closure respectively.

For example, if D and E are cpo's with least elements and $\mathrm{fst} : D \times E \to D$ is given by $\mathrm{fst}(x, y) = x$ then fst is a continuous projection with $\mathrm{fst}' : D \to D \times E$ given by $\mathrm{fst}(x) = (x, \bot_E)$ as corresponding embedding. Similarly, snd is also a continuous projection. (It is *not* true, however, that if $f : F \to D$ and $g : F \to E$ are continuous projections then

$\langle f, g\rangle : F \to D \times E$ is a projection.)

In an arbitrary category, an arrow $r : B \to A$ is said to be a *retraction* if there is an arrow $r' : A \to B$ (called a *section*) such that $r \circ r' = \mathrm{id}_A$. If there is a retraction $r : B \to A$ then A is said to be a *retract* of B. Note that for an adjunction $\langle p, q\rangle : B \xrightarrow{\text{adj}} A$, $\langle p, q\rangle$ is a pe-pair if and only if p is a retraction with q as a section and $\langle p, q\rangle$ is a closure if and only if q is a retraction with p as a section.

If $\langle p, q\rangle : E \xrightarrow{\text{adj}} D$ is an adjunction between cpo's D and E such that p is continuous then q sends finite elements of D to finite elements of E. To see this, suppose x is finite in D and M is a directed subset of E. If $q(x) \sqsubseteq \bigsqcup M$ then $x \sqsubseteq p(q(x)) \sqsubseteq p(\bigsqcup M) = \bigsqcup p(M)$. Since x is finite, there is a $y \in M$ with $x \sqsubseteq p(y)$ so $q(x) \sqsubseteq q(p(y)) \sqsubseteq y$. We say that the adjunction $\langle p, q\rangle$ is continuous if p and q are. In proving that the adjunction is continuous, it is not necessary to check that q is continuous, however, because a lower adjoint is *always* continuous. To see this, suppose $\langle p, q\rangle : E \xrightarrow{\text{adj}} D$ and $M \subseteq D$ is directed. Since q is monotone, we know that $q(\bigsqcup M) \sqsupseteq \bigsqcup q(M)$. Now,

$$\begin{aligned} \textstyle\bigsqcup q(M) &\sqsupseteq q \circ p(\textstyle\bigsqcup q(M)) \\ &\sqsupseteq q(\textstyle\bigsqcup p \circ q(M)) && \text{by the monotonicity of } p \\ &\sqsupseteq q(\textstyle\bigsqcup M). \end{aligned}$$

Hence $q(\bigsqcup M) = \bigsqcup q(M)$ and q is therefore continuous. We refer to a continuous upper adjoint as a *homomorphism* of cpo's.

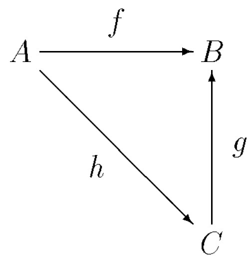
In some places an adjunction is defined to be a pair of monotone functions $p : B \to A$ and $q : A \to B$ such that $p(Y) \sqsupseteq X$ if and only if $Y \sqsupseteq q(X)$. Note, in particular, that p uniquely determines q and, conversely, q uniquely determines p. It is easy to show that this definition is equivalent to the one given above. Some more properties of adjunctions which will be needed are summarized in the following lemma. The book [Gierz *et. al.* 1980] contains proofs of these facts as well as many other details about adjunctions between posets.

Lemma 17 *If* $\langle p, q\rangle : B \xrightarrow{\text{adj}} A$ *and* $\langle p', q'\rangle : C \xrightarrow{\text{adj}} B$ *then*

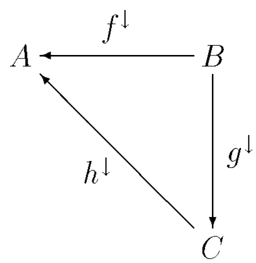
1. $\langle p \circ p', q' \circ q\rangle : C \xrightarrow{\text{adj}} A$ *(and similar results hold for pe-pairs and closures);*
2. $p \circ q \circ p = p$ *and* $q \circ p \circ q = q$*;*
3. $im(p \circ q) = im(p) \cong im(q) = im(q \circ p)$. □

We let $\mathbf{CPO}^{\uparrow}$ be the category of cpo's with continuous upper adjoints (homomorphisms) as arrows. The sub-category of cpo's with continuous projections is denoted $\mathbf{CPO}^{P}$. The dual categories are $\mathbf{CPO}^{\downarrow}$ and $\mathbf{CPO}^{E}$ respectively. Other variants such as $\mathbf{ALG}^{E}$ and $\omega\mathbf{ALG}^{\downarrow}$ have the obvious meanings. If D and E are cpo's and $\langle p, q\rangle : E \xrightarrow{\text{adj}} D$ then we set $p^{\downarrow} = q$ and $q^{\uparrow} = p$. If p is a projection we may write p^{E} rather than $p^{\downarrow}$ and q^{P} rather than $q^{\uparrow}$.

An especially important feature of adjunctions is that of *duality*. A diagram



in $\mathbf{CPO}^{\uparrow}$ commutes if and only if the dual diagram



in $\mathbf{CPO}^{\downarrow}$ commutes. As far as the solution of recursive domian equations goes, this gives rise to a noteworthy equivalence between solutions constructed in $\mathbf{CPO}^{\uparrow}$ and solutions constructed in $\mathbf{CPO}^{\downarrow}$.

Definition: If D is an cpo then a continuous function $f : D \to D$ is said to be a *deflation* if $f \circ f = f \sqsubseteq \mathrm{id}_D$. An *algebraic deflation* is a deflation whose image is an algebraic cpo. A *finite deflation* is a deflation whose image is finite. An *inflation* is a continuous $f : D \to D$ such that $f \circ f = f \sqsupseteq \mathrm{id}_D$. □

Images of inflations and deflations are always cpo's. For if D is a cpo and $r : D \to D$ is idempotent and continuous then for any directed $M \subseteq im(r)$, $\bigsqcup M = \bigsqcup r(M) = r(\bigsqcup M) \in im(r)$. Deflations and inflations are closely related to adjoint pairs. If $\langle p, q \rangle : E \xrightarrow{\mathrm{adj}} D$ then $q \circ p$ is a deflation on E and $p \circ q$ is an inflation on D. On the other hand, if $p : E \to E$ is a deflation and $D = im(p)$ then the corestriction $p^\circ : E \to D$ of p to its image is a projection with the inclusion map from D into E as its corresponding embedding. This shows that $\mathbf{B}[D] \subseteq \mathbf{B}[E]$ since an embedding sends finite elements to finite elements. But $D \cap \mathbf{B}[E] \subseteq \mathbf{B}[D]$ so we must have $\mathbf{B}[D] = D \cap \mathbf{B}[E]$. Now, if $q : D \to D$ is an inflation and D is an algebraic cpo then $im(q) = E$ is also algebraic. For if $x \in E$ and $M = \{y \in \mathbf{B}[D] \mid y \sqsubseteq x\}$ then $x = \bigsqcup M$ since D is algebraic. But $\bigsqcup q(M) = q(\bigsqcup M) = q(x) = x$ and $q(M) \subseteq \mathbf{B}[E]$ since q° is a lower adjoint (with the inclusion map from E into D and the corresponding upper adjoint).

Definition: If A is a pre-order then we denote by $N(A)$ the set of normal substructures of A, where if $B, C \in N(A)$ then $C \vdash_{N(A)} B$ if an only if for every $X \in C$ there is a $Y \in B$ such that $X \vdash_A Y$. □

Proposition 18 *Let A be a poset. Then $N(A)$ is a cpo. If A has property* m *then it is an algebraic lattice. If A is a Plotkin poset then $N(A)$ is a locally finite algebraic lattice (i.e. $\{x_0 \in \mathbf{B}[N(A)] \mid x \sqsubseteq x_0 \sqsubseteq y\}$ is finite for each $x, y \in \mathbf{B}[N(A)]$).*

Proof. Since A is a poset the order on $N(A)$ is just subset inclusion. Suppose $M \subseteq N(A)$ is directed and $X \in A$. If $u \subseteq \downarrow X \cap \bigcup M$ is finite then $u \subseteq B$ for some $B \in M$. Since $B \lhd A$ there is an $X' \in B$ such that $X \vdash X' \vdash u$. Hence $\bigcup M \in N(A)$. Obviously, $\bigcup M$ is a least upper bound for M. Let $\mathcal{P}(A)$ be the set of subsets of A, ordered by $\subseteq$. If A has property m then $\mathcal{U}^* : \mathcal{P}(A) \to \mathcal{P}(A)$ is an inflation with image $N(A)$. Since $\mathcal{P}(A)$ is an algebraic lattice whose finite elements are the finite subsets of A, it follows that $N(A)$ is an algebraic lattice and has a basis of finite elements of the form $\mathcal{U}^*(u)$ where $u \subseteq A$ is finite. If A is a Plotkin order then $\mathcal{U}^*(u)$ is finite for each finite u, so there can be only finitely many elements below it. Hence $N(A)$ is locally finite. □

Lemma 19 *Let D be a cpo. If $p : D \to D$ is a deflation then $im(p) \lhd D$. Moreover, for any pair $p, q : D \to D$ of deflations, $q \sqsubseteq p$ if and only if $im(q) \subseteq im(p)$.*

Proof. Suppose $y, z \in M = im(p) \cap \downarrow x$. Then $x \sqsupseteq p(x) \sqsupseteq y, z$, so M is directed. Suppose $p, p' : D \to D$ are deflations. If $p' \sqsubseteq p$ then for each x, $p'(x) \sqsubseteq p(p'(x)) \sqsubseteq p'(p'(x)) = p'(x)$. So $p'(x) = p(p'(x))$ and therefore $im(p') \subseteq im(p)$. On the other hand, if $im(p') \subseteq im(p)$ and $x \in D$ then $p'(x) = p(p'(x)) \sqsubseteq p(x)$ so $p' \sqsubseteq p$. □

Proposition 20 *Let D be an algebraic cpo and suppose $p : D \to D$ is an algebraic deflation. Then $\mathbf{B}[im(p)] = im(p) \cap \mathbf{B}[D] \lhd \mathbf{B}[D]$. On the other hand, if $A \lhd \mathbf{B}[D]$ then $E = \{\bigsqcup M \mid M \subseteq A$ is directed$\}$ is the image of an algebraic deflation on D. Hence there is an isomorphism between $N(\mathbf{B}[D])$ and the poset of algebraic deflations on D.*

Proof. If $p : D \to D$ is an algebraic deflation then $im(p) \lhd D$ by Lemma 19. Since $im(p)$ is algebriac, $\mathbf{B}[im(p)] \lhd im(p)$ so $\mathbf{B}[im(p)] \lhd D$. But $\mathbf{B}[im(p)] \subseteq \mathbf{B}[D]$ so $\mathbf{B}[im(p)] = im(p) \cup \mathbf{B}[D] \lhd \mathbf{B}[D]$.

Now, suppose $A \lhd \mathbf{B}[D]$ and $E = \{\bigsqcup M \mid M \subseteq A \text{ is directed}\}$. For each $x \in D$, define $p(x) = \bigsqcup\{y \in A \mid x \sqsupseteq y\}$. By the assumption on A, this is a well-defined surjection onto E. To see that p is continuous, suppose $M \subseteq D$ is directed. Then

$$\begin{aligned}
\bigsqcup p(M) &= \bigsqcup\{p(x) \mid x \in M\} \\
&= \bigsqcup\{\bigsqcup\{y \in A \mid x \sqsupseteq y\} \mid x \in M\} \\
&= \bigsqcup\{y \in A \mid y \in A \cup \downarrow M\} \\
&= \bigsqcup\{y \in A \mid \bigsqcup M \sqsupseteq y\} \\
&= p(\bigsqcup M).
\end{aligned}$$

That $p \sqsubseteq \mathrm{id}$ is immediate from the definition of p. Moreover,

$$\begin{aligned}
p \circ p(x) &= \bigsqcup\{y \in A \mid p(x) \sqsupseteq y\} \\
&= \bigsqcup\{y \in A \mid \bigsqcup\{z \in A \mid x \sqsupseteq z\} \sqsupseteq y\} \\
&= \bigsqcup\{y \in A \mid x \sqsupseteq y\} \\
&= p(x).
\end{aligned}$$

That E is algebraic follows from its definition so p is an algebraic deflation. By Lemma 19 the correspondence $p \mapsto \mathbf{B}[im(p)]$ is therefore an isomorphism between the poset of algebraic deflations on D and $N(\mathbf{B}[D])$. □

Corollary 21 *Let D be an algebraic cpo and suppose D' is the poset of algebraic deflations on D. Then D' is a cpo. If $\mathbf{B}[D]$ has property m then D' is algebraic and if $\mathbf{B}[D]$ is a Plotkin order then D' is locally finite.*

Proof. This follows immediately from 18 and 20. □

We next look at the relationship between normal substructures of pre-orders and pe-pairs from the point of view of approximable relations. We thereby generalize the theory exposited in [Scott 1981b] to the category of algebraic cpo's. These results will be used shortly to derive a universal domain technique for the Plotkin orders. The most immediate application, however, is to study the existence of limits in various categories of algebraic cpo's. Let A and B be pre-orders. Write $A \mathrel{\underset{\sim}{\lhd}} B$ if there is an $A' \lhd B$ such that $A \cong A'$ (in the category with approximable relations as arrows). We have the following:

Theorem 22 *Let A and B be pre-orders.*

1. *Suppose $A \lhd B$ and $\vdash$ is the order relation on $B \times B$. If $p = (B \times A) \cap \vdash$ and $q = (A \times B) \cap \vdash$ then p, q are approximable relations, $p \circ q = \mathrm{id}_A$ and $q \circ p \subseteq \mathrm{id}_B$. In other words $\langle |p|, |q| \rangle : |B| \xrightarrow{\mathrm{pe}} |A|$.*

2. *Conversely, if $\langle |p|, |q| \rangle : |B| \xrightarrow{\mathrm{pe}} |A|$ for approximable relations p and q then $A \mathrel{\underset{\sim}{\lhd}} B$. In particular,*
$$A \cong A' = \{Y \in B \mid Y\ (q \circ p)\ Y\} \lhd B.$$

Proof. The proof of 1 is a straight-forward verification. To prove 2, we begin by showing that $A' \lhd B$. Suppose $u \subseteq A'$ is finite and $Z \vdash u$. For each $X \in u$, there is an $X' \in A$ such that $X\ p\ X'\ q\ X$. Let $v = \{X' \mid X \in u\}$. Then $Z\ p\ X'$ for each $X' \in v$ so there is a $Y \in A$ such that $Z\ p\ Y \vdash v$. Now, $Y\ p \circ q\ Y$ so there is a $Z' \in B$ such that $Y\ q\ Z'\ p\ Y$. But then $Z'\ p\ Y\ q\ Z'$ so $Z' \in A'$. If $X \in u$ then $Y \vdash X'$ so $Y\ q\ X$. Since $Z'\ p\ Y$ we get $Z'\ q \circ p\ X$ and therefore $Z' \vdash X$. Moreover, $Z\ p\ Y\ q\ Z'$ so $Z \vdash Z'$.

Let $p' = p \cap (A' \times A)$ and $q' = q \cap (A \times A')$. That p' is approximable follows immediately from the approximability of p. If $X \in A$ and $X\ q'\ Y, Y'$ for $Y, Y' \in A'$ then $X\ q\ Z$ for some $Z \in B$ such that $Z \vdash Y, Y'$. Since $A' \lhd B$, there is a $Z' \in A'$ such that $Z \vdash Z' \vdash Y, Y'$. Hence $X\ q'\ Z'$. The other conditions are easy to check. Now, suppose $X \in A$. Then $X\ p \circ q\ X$ so $X\ q\ Y\ p\ X$ for some $Y \in B$. But then $Y \in A'$ so $X\ p' \circ q'\ X$. Since $p' \circ q' \subseteq \mathrm{id}_A$, we conclude that $p' \circ q' = \mathrm{id}_A$. Suppose on the other hand, that $Y \in A'$. Then, by definition, $Y\ q \circ p\ Y$. Since $q' \circ p' \subseteq \mathrm{id}_{A'}$ we must have $q' \circ p' = \mathrm{id}_{A'}$. This proves the desired isomorphism. □

Theorem 23 *Suppose A is a pre-order and $f : A \to A$ is an approximable relation. Then the following are equivalent:*

1. *$|f|$ is an algebraic deflation.*

2. $f \circ f = f \subseteq \mathrm{id}_A$ *and whenever* $X\ f\ Z$*, then* $X \vdash Y\ f\ Y \vdash Z$ *for some* $Y \in A$*.*

Proof. (1) $\Rightarrow$ (2). Suppose $X\ f\ Z$. Then $Z \in |f|(\downarrow X)$ and since $im(|f|)$ is algebraic there is a finite $x \in im(|f|)$ such that $Z \in x \subseteq |f|(\downarrow X)$. But x is finite in $|A|$ so $x = \downarrow Y$ for some Y. This Y has the property in the conclusion of (2).

(2) $\Rightarrow$ (1). Certainly, if (2) holds then $|f|$ is a deflation. To see that it has an algebraic image, note that if $X\ f\ X$ then $\downarrow X = |f|(\downarrow X)$ so $\downarrow X$ is a finite element of $im(|f|)$. If $x \in |A|$ then

$$\begin{aligned} |f|(x) &= \{Z \mid X\ f\ Z \text{ some } X \in x\} \\ &= \{Z \mid X \vdash Y\ f\ Y \vdash Z \text{ some } X \in x \text{ and some } Y\} \\ &= \bigcup\{\downarrow Y \mid Y \in x \text{ and } Y\ f\ Y\}. \end{aligned}$$

To see that this set is directed, suppose $X\ f\ X$ and $Y\ f\ Y$. If $Z\ f\ X, Y$ then $Z\ f\ Z' \vdash X, Y$ for some Z'. Hence $Z \vdash W\ f\ W \vdash Z' \vdash X, Y$ for some W. We conclude that $im(|f|)$ is algebraic. □

Definition: A class $\mathcal{K}$ of cpo's is *closed under homomorphic images* if for every pair D, E of members of $\mathcal{K}$ and homomorphism $p : E \to D$, the image of p is a member of $\mathcal{K}$. A class $\mathcal{K}$ of posets is *closed under normal substructures* if whenever $B \trianglelefteq A$ and A is in $\mathcal{K}$ then B is in $\mathcal{K}$. □

Theorem 24 *If* $\mathcal{K}$ *is a class of posets which is closed under normal substructures then* $\mathbf{Id}_{\mathcal{K}}$ *is closed under homomorphic images.*

Proof. Suppose $\langle p, q\rangle : E \xrightarrow{\text{adj}} D$ is continuous and D, E are in $\mathbf{Id}_{\mathcal{K}}$. Let $D' = im(p)$ and $E' = im(q)$. Note, in particular, that D' is the image of an inflation and is therefore algebraic. Furthermore, $E' \cong D'$ by 173. and $\mathbf{B}[E'] \triangleleft \mathbf{B}[E]$ by 20. Since $\mathcal{K}$ is closed under normal substructures, $\mathbf{B}[E']$ is in $\mathcal{K}$ so $D' \cong |\mathbf{B}[E']|$ is an object in $\mathbf{Id}_{\mathcal{K}}$. □

3.2 Inverse limits

In this section we investigate further the categorical importance of the Plotkin orders. To this end we introduce the inverse limit construction and show that ideal completions of Plotkin orders are closely related to certain inverse limits of finite posets. The results below generalize those in the orginal treatment by Plotkin [1976] and the results on adjunctions in [Niño 1981].

Let $\langle I, \geq\rangle$ be a directed, transitive and reflexive ordering. An *inverse system* $\langle D_i, d_{ij}\rangle$ (in order type $\langle I, \geq\rangle$) over a category $\mathbf{C}$ is a collection $\{D_i \mid i \in I\}$ of $\mathbf{C}$-objects together with a set of arrows $d_{ij} : D_i \to D_j$ where $i, j \in I$ and $i \geq j$ If $i, j, k \in I$ and $i \geq j \geq k$ then the maps d_{jk}, d_{ij}, d_{ik} are required to satisfy the equation

$$d_{jk} \circ d_{ij} = d_{ik}, \qquad (*)$$

and for each i, $d_{ii} = \mathrm{id}_{D_i}$. Let $\Delta = \langle D_i, d_{ij}\rangle$ be an inverse system in a category $\mathbf{C}$. A *cone* $\mu : D \to \Delta$ is an object D called the *vertex* of the cone together with a set of arrows $\mu_i : D \to D_i$ such that for each $i \geq j$, $d_{ij} \circ \mu_i = \mu_j$. μ is said to be a *limiting* (or *initial*) cone if for any cone $\nu : D' \to \Delta$ there is a unique *mediating arrow* $f : D' \to D$ such that for each i, $\mu_i \circ f = \nu_i$.

Let $\Delta = \langle D_i, d_{ij} \rangle$ be an inverse system in **CPO**. The *inverse limit* of Δ is the partial order $\langle D_*, \sqsubseteq \rangle$ where the elements of D_* are functions

$$x : I \to \bigcup_{i \in I} D_i$$

such that for each $i, j \in I$ with $i \geq j$, $x(i) = x_i \in D_i$, and $d_{ij}(x_i) = x_j$. The ordering is determined termwise, *i.e.* if $x, y \in D_*$ then $x \sqsubseteq y$ if and only if $x_i \sqsubseteq_{D_i} y_i$ for each $i \in I$.

Theorem 25 *Suppose $\Delta = \langle D_i, d_{ij} \rangle$ is an inverse system of order type I in* **CPO***. For each i, let $d_{*i} : D_* \to D_i$ by $d_{*i}(x) = x_i$. Then*

$$d = \langle d_{*i} \rangle : D_* \to D_i$$

is a limiting cone in **CPO***.*

Proof. The fact that the maps d_{*i} are continuous and d is a cone is immediate from the definitions. To see that it is a limiting cone, suppose $\langle D, f_i \rangle$ is a **CPO** cone over Δ. Let $f : D \to D_*$ by $f(x) = \langle f_i(x) \rangle_{i \in I}$. Now $\langle f_i(x) \rangle_{i \in I}$ is in D_* because $d_{ij} \circ f_i = f_j$ for each $i \geq j$. If $M \subseteq D$ is directed, then $f(\bigsqcup M) = \langle f_i(\bigsqcup M) \rangle_{i \in I} = \langle \bigsqcup f_i(M) \rangle_{i \in I} = \bigsqcup f(M)$ so f is continuous. By definition, $d_{*i} \circ f = f_i$ for each i and this uniquely determines f. □

The vertex of a limiting cone is unique (up to isomorphism) so we are justified in denoting the inverse limit of a **CPO** system Δ by $\varprojlim \Delta$. (Although it is easier to write D_* for $\varprojlim \langle D_i, d_{ij} \rangle$ when there is no chance for confusion.) The theorem also reinforces the legitimacy of the term "inverse *limit*". We will be especially interested in inverse systems of cpo's where the functions d_{ij}, $i \geq j$, are continuous upper adjoints. When $i \geq j$ it is useful to define $d_{ji} : D_j \to D_i$ to be the lower adjoint corresponding to d_{ij}. It follows from Lemma 17 that d_{ji} is uniquely determined by d_{ij} and equation $(*)$ holds even if $k \geq j \geq i$.

Lemma 26 *Let $\langle D_i, d_{ij} \rangle$ be a* $\mathbf{CPO}^{\uparrow}$ *inverse system. If $l \geq k \geq i, j$ then*

1. *$d_{kj} \circ d_{ik} \sqsubseteq d_{lj} \circ d_{il}$, and*
2. *$d_{kj} \circ d_{ik}(x_i) \sqsubseteq x_j$ for each $x \in D_*$.*

Proof. For part (1),

$$\begin{aligned} d_{lj} \circ d_{il} &= (d_{kj} \circ d_{lk}) \circ (d_{kl} \circ d_{ik}) \\ &= d_{kj} \circ (d_{lk} \circ d_{kl}) \circ d_{ik} \\ &\sqsubseteq d_{kj} \circ d_{ik}. \end{aligned}$$

For part (2),

$$\begin{aligned} d_{kj} \circ d_{ik}(x_i) &= d_{kj} \circ d_{ik} \circ d_{ki}(x_k) \\ &\sqsubseteq d_{kj}(x_k) \\ &= x_j. \end{aligned}$$

□

Fix a $\mathbf{CPO}^{\uparrow}$ inverse system $\langle D_i, d_{ij}\rangle$. We wish to define $d_{i*} : D_i \to D_*$ by setting

$$(d_{i*}(x))_j = \bigsqcup_{k \geq i,j} d_{kj} \circ d_{ik}(x)$$

for each $x \in D_i$ and $j \in I$. By Lemma 261., the set on the right is directed so the supremum in question exists in D_j. Suppose that $j \geq l$. Then

$$\begin{aligned} d_{jl}((d_{i*}(x))_j) &= \bigsqcup_{k \geq i,j} d_{jl} \circ d_{kj} \circ d_{ik}(x) \\ &= \bigsqcup_{k \geq l,j} d_{kl} \circ d_{ik}(x) \\ &= (d_{i*}(x))_l \end{aligned}$$

so $d_{i*}(x) \in D_*$. It is easy to see that d_{i*} is monotone. If $x \in D_i$ then

$$d_{*i}(d_{i*}(x)) = \bigsqcup_{k \geq i,j} d_{ki} \circ d_{ik}(x) \sqsupseteq x$$

so $d_{*i} \circ d_{i*} \sqsupseteq \mathrm{id}_{D_i}$. If $x \in D_*$ then

$$(d_{i*} \circ d_{*i}(x))_j = \bigsqcup_{k \geq i,j} d_{kj} \circ d_{ik}(x_i) \sqsubseteq x_j$$

by Lemma 262. Hence $d_{i*} \circ d_{*i} \sqsubseteq \mathrm{id}_{D_*}$. So $\langle d_{*i}, d_{i*}\rangle$ is a continuous adjunction. Thus $d : D_* \to \Delta$ where d is the set of arrows d_{*i} is a cone over $\langle D_i, d_{ij}\rangle$ in $\mathbf{CPO}^{\uparrow}$. Next we show that this cone is also limiting in $\mathbf{CPO}^{\uparrow}$.

Theorem 27 *Let $\Delta = \langle D_i, d_{ij}\rangle$ be an inverse system in $\mathbf{CPO}^{\uparrow}$. Then then the cone*

$$d = \langle d_{*i}\rangle : D_* \to D_i$$

which is a limiting cone in $\mathbf{CPO}$ *is also limiting in* $\mathbf{CPO}^{\uparrow}$

Proof. Suppose $\langle p_i\rangle_{i\in I} : D \to \Delta$ is a cone in $\mathbf{CPO}^{\uparrow}$ where for each i,

$$\langle p_i, q_i\rangle : D \xrightarrow{\text{adj}} D_i.$$

If $i \geq j$ then $q_i \circ d_{ji} = q_j$ so $q_i \circ d_{*i} \sqsupseteq q_i \circ d_{ji} \circ d_{ij} \circ d_{*i} = q_j \circ d_{*j}$. We may therefore define a monotone map $q : D_* \to D$ by $q = \bigsqcup_i q_i \circ d_{*i}$. As in the proof of Theorem 25 we define $p : D \to D_*$ by setting $d_{*i} \circ p = p_i$ for each i. Now,

$$d_{*i} \circ p \circ q = p_i \circ (\bigsqcup_j q_j \circ d_{*j}) \sqsupseteq p_i \circ q_i \circ d_{*i} \sqsupseteq d_{*i}$$

so $p \circ q \sqsupseteq \mathrm{id}_{D_*}$. On the other hand, $q \circ p = \bigsqcup_i q_i \circ p_i \sqsubseteq \mathrm{id}_D$. Hence q is a lower adjoint for p and d is a limiting cone. □

Recall that a transitive and reflexive ordering $\langle I, \geq\rangle$ is *filtered* if I^{op} is directed, *i.e.* for each $i, j \in I$ there is a $k \in I$ such that $k \leq i, j$. An *direct system* $\langle D_i, d_{ij}\rangle$ (in order type

$\langle I, \geq \rangle$) over a category $\mathbf{C}$ is a collection $\{D_i \mid i \in I\}$ of $\mathbf{C}$-objects together with a set of arrows $d_{ij} : D_i \to D_j$ where $i, j \in I$ and $j \geq i$. If $i, j, k \in I$ and $k \geq j \geq i$ then the maps d_{jk}, d_{ij}, d_{ik} are required to satisfy the equation

$$d_{jk} \circ d_{ij} = d_{ik}, \tag{$*$}$$

and for each i, $d_{ii} = \mathrm{id}_{D_i}$. Let $\Delta = \langle D_i, d_{ij} \rangle$ be an direct system in a category $\mathbf{C}$. A *cocone* $\mu : \Delta \to D$ is an object D called the *vertex* of the cone together with a set of arrows $\mu_i : D_i \to D$ such that for each $j \geq i$, $\mu_i \circ d_{ji} = \mu_j$. μ is said to be a *colimiting* cocone if for any cocone $\nu : \Delta \to D'$ there is a unique *mediating arrow* $f : D \to D'$ such that for each i, $f \circ \mu_i = \nu_i$.

Theorem 28 *(Limit/colimit duality.) Let $\Delta = \langle D_i, d_{ij} \rangle_{i \geq j}$ be an inverse system in $\mathbf{CPO}^{\uparrow}$. Then the direct system*

$$\Delta^{\downarrow} = \langle D_i, d_{ji} \rangle_{i \geq j}$$

has a colimit $\varinjlim \Delta^{\downarrow}$ in $\mathbf{CPO}^{\downarrow}$ and

$$\varinjlim \Delta^{\downarrow} \cong \varprojlim \Delta.$$

Proof. Let $D_* = \varprojlim \Delta$. We claim that D_* is a colimit for $\Delta^{\downarrow}$. For each $i \in I$ there is a lower adjoint $d_{i*} : D_i \to D_*$. By the duality between $\mathbf{CPO}^{\uparrow}$ and $\mathbf{CPO}^{\downarrow}$, $\langle d_{i*} \rangle_{i \in I} : D_i \to D_*$ is a cocone over $\Delta^{\downarrow}$. Suppose $\mu_i : D_i \to E$ is another cocone. Then $\mu_i^{\uparrow} : E \to D_i$ is a $\mathbf{CPO}^{\uparrow}$ cone over Δ so there is a unique adjoint $\langle p, q \rangle : E \xrightarrow{\text{adj}} D_*$ such that $d_{*i} \circ p = \mu_i^{\uparrow}$ for each i. Hence, for each i, $q \circ d_{i*} = \mu_i$. If $\langle p', q' \rangle : E \xrightarrow{\text{adj}} D_*$ and $q' \circ d_{i*} = \mu_i$ for each i then $d_{*i} \circ p' = \mu_i^{\uparrow}$ for each i so $p' = p$. Since q' is uniquely determined by p' we have $q' = q$. Hence q is the unique lower adjoint such that $q \circ d_{i*} = \mu_i$ for each i. We conclude that the cocone $\langle d_{i*} \rangle$ is colimiting and D_* is therefore a colimit. □

Lemma 29 *$\bigsqcup_i d_{i*} \circ d_{*i}$ is the identity function on D_*.*

Proof. Suppose $j \geq i$, then $d_{i*} \circ d_{*i} = (d_{j*} \circ d_{ij}) \circ (d_{ji} \circ d_{*j}) \sqsubseteq d_{j*} \circ d_{*j}$. Since I is directed it follows that $\{d_{i*} \circ d_{*i} \mid i \in I\}$ is directed. Let $x \in D_*$ and let $j \in I$. Then

$$d_{*j} \circ (\bigsqcup_i d_{i*} \circ d_{*i}) = d_{*j} \circ d_{j*} \circ d_{*j} = d_{*j}.$$

□

When the arrows in the inverse system Δ are projections then so are the maps d_{*i}. In fact, in this case μ is a $\mathbf{CPO}^P$ limit. This follows from Lemma 29 together with the following:

Lemma 30 *Let $\mu : D \to \Delta$ be a cone in $\mathbf{CPO}^P$ where Δ is a $\mathbf{CPO}^P$ inverse system in order type I. Then $M = \{\mu_i^E \circ \mu_i \mid i \in I\}$ is directed and $\bigsqcup M = \mathrm{id}_D$ if and only if μ is a limiting cone.* □

Proof of the Lemma for $I = \omega$ can be found in [Smyth and Plotkin 1982] or [Brookes 1984]. Niño[1981] observes that the lemma holds even if Δ is only a $\mathbf{CPO}^{\uparrow}$ inverse system.
Definition: Let $M = \{A_i \mid i \in I\}$ be a set of posets indexed by a directed poset I such that $j \leq i$ implies $A_j \lhd A_i$. We refer to such a collection M as *normal directed system* (in order type I). A class $\mathcal{K}$ of posets is *closed under (countable) normal directed unions* if whenever M is a (countable) normal directed system of posets isomorphic to posets from $\mathcal{K}$ then $\bigcup M$ is isomorphic to a poset in $\mathcal{K}$. □

Lemma 31 *Let A be a pre-order and $\{A_i \mid i \in I\}$ a set of normal substructures of A indexed by a directed poset I. Suppose, moreover, that $\bigcup_{i \in I} A_i = A$ and $A_j \subseteq A_i$ whenever $j \leq i$. For each $i \geq j$, let $a_{ij} = (A_i \times A_j) \cap \vdash$. Then $\Delta = \langle |A_i|, |a_{ij}| \rangle$ is a $\mathbf{CPO}^P$ inverse system with $|A| \cong \varprojlim \Delta$.*

Proof. For each i, let $a_i = (A \times A_i) \cap \vdash$ and $a_i^E = (A_i \times A) \cap \vdash$. By the various results about ideal completions together with Theorem 22, we know that the relations a_{ij}, a_i, a_i^E are approximable, $\Delta = \langle |A_i|, |a_{ij}| \rangle$ is an inverse system in $\mathbf{CPO}^P$ and $\mu : |A| \to \Delta$ is a $\mathbf{CPO}^P$ cone if μ is the set of functions $|a_i|$. Since $\bigcup_{i \in I} A_i = A$ we also have $\bigcup_{i \in I} a_i^E \circ a_i = \mathrm{id}_A$ so μ is a limiting cone by Theorem 30. Hence $|A| \cong \varprojlim \Delta$. □

Theorem 32 *Let $\Delta = \langle D_i, d_{ij} \rangle$ be a $\mathbf{ALG}^P$ inverse system. Then $D_* = \varprojlim \Delta$ is algebraic and $\mathbf{B}[D_*] = \bigcup_{i \in I} d_{i*}(\mathbf{B}[D_i])$. Hence, if $\mathcal{K}$ is a class of posets which is closed under (countable) directed unions then $\mathbf{Id}_{\mathcal{K}}^P$ has (countable) inverse limits.*

Proof. By Lemma 20 the posets $d_{i*}(\mathbf{B}[D_i])$ form a normal directed system. If $A = \bigcup_{i \in I} d_{i*}(\mathbf{B}[D_i])$ then by Lemma 31, $|A| \cong D_*$ so D_* is algebraic and A must be its basis. Now, if Δ is an $\mathbf{Id}_{\mathcal{K}}^P$ inverse system then $\mathbf{B}[D_*]$ is isomorphic to a poset in $\mathcal{K}$ since $\mathcal{K}$ is closed under normal directed unions. Hence $D_* \cong |\mathbf{B}[D_*]|$ must be an object in $\mathbf{Id}_{\mathcal{K}}$ so $\mathbf{Id}_{\mathcal{K}}^P$ has inverse limits. □

Let $M = \{p_i \mid i \in I\}$ be the set of deflations on a cpo D where I is a directed poset such that $i \geq j$ implies $p_i \sqsupseteq p_j$. For each i, let $D_i = im(p_i)$. If $i \geq j$, let $p_{ij} : D_i \to D_j$ be the restriction of p_j° to D_i. Now, p_{ij} is a projection with the inclusion map $incl_{ji} : D_j \hookrightarrow D_i$ as the corresponding embedding. Also, p_i° is a projection and the inclusion $incl_i : D_i \hookrightarrow D$ is the corresponding embedding. If $i \geq j \geq k$ then $incl_i \circ incl_{ji} = incl_j$ and $incl_i \circ incl_{ji} = incl_j$ so by Lemma 17, we have $p_{jk} \circ p_{ij} = p_{ik}$ and $p_{ij} \circ p_i = p_j$. Hence $\Delta_M = \langle D_i, p_{ij} \rangle$ is a $\mathbf{CPO}^P$ inverse system and $\mu_M = \langle p_i^\circ \rangle_{i \in I} : D \to \Delta_M$ is a cone. Let us refer to μ_M as the *cone of projections determined by M*. By Lemma 30 we have the following:

Lemma 33 *Suppose $\mu : D \to \Delta$ is a $\mathbf{CPO}^{\uparrow}$ cone over an inverse system Δ in order type I and let $M = \{\mu_i^{\downarrow} \circ \mu_i \mid i \in I\}$. Let $\mu_M : D \to \Delta_M$ be the cone of projections determined by M. Then μ_M is initial if and only if $\bigsqcup M = \mathrm{id}_D$.* □

The following consequence of Lemma 33 describes sufficient conditions for a class $\mathcal{K}$ to have $\mathbf{CPO}^{\uparrow}$ inverse limits.

Theorem 34 *Let $\mathbf{C}$ be a full sub-category of $\mathbf{CPO}$. If $\mathbf{C}$ is closed under homomorphic images and $\mathbf{C}^P$ has inverse limits then $\mathbf{C}^{\uparrow}$ has limits for inverse systems.*

Proof. Suppose $\langle D_i, d_{ij}\rangle$ is an inverse limit in $\mathbf{C}^{\uparrow}$. Let $D = \varprojlim\langle D_i, d_{ij}\rangle$ and define deflations $d_{i*} \circ d_{*i} = p_i : D \to D$. Let $M = \{p_i \mid i \in I\}$ and suppose $\mu_M : D \to \Delta_M$ is the cone of projections generated by M. Since $\mathbf{C}$ is closed under homomorphic images, this cone lies in $\mathbf{C}^P$. To see that it is initial, by Lemma 33 it suffices to show that $\bigsqcup M = \mathrm{id}_D$. But this is exactly what is asserted by Lemma 29. Since $\mathbf{C}^P$ is closed under inverse limits it follows that D is in $\mathbf{C}$ so $\mathbf{C}^{\uparrow}$ has limits for inverse systems. □

Corollary 35 *Let $\mathcal{K}$ be a class of posets. If $\mathcal{K}$ is closed under (countable) normal directed unions and normal substructures then $\mathbf{Id}_{\mathcal{K}}$ has (countable) $\mathbf{CPO}^{\uparrow}$ inverse limits.*

Proof. This follows from Theorem 24 together with Theorems 32 and 34. □

Proposition 36 *If $\mathcal{K}$ is any one of the following classes of posets then $\mathbf{Id}^{\uparrow}_{\mathcal{K}}$ has limits for inverse systems:*

1. *posets with property* m*,*
2. *posets with property* M*,*
3. *Plotkin posets.*

Moreover, for each of the corresponding classes of countable posets $\mathbf{Id}^{\uparrow}_{\mathcal{K}}$ has countable inverse limits.

Proof. In light of Lemma 11 and Corollary 35 it suffices to show that each of the three given classes is closed under normal directed unions. Suppose that $A = \bigcup M$ where M is a normal directed system of posets having property M. If $u \subseteq A$ is finite then $u \subseteq B$ for some $B \in M$. Since B has property M, u has a complete set of minimal upper bounds in B. but this complete set is also complete in A since $B \lhd A$. Hence A has property M. The argument for property m is essentially the same. If each of the posets in M is a Plotkin poset and $u \subseteq A$ then $u \subseteq B$ for some $B \in M$ and $\mathcal{U}^*_B(u) \lhd A$. Thus A is a Plotkin order. □
Definition: A *profinite domain* is a cpo which is isomorphic to the limit of a $\mathbf{CPO}^P$ inverse system of finite posets. Define $\mathbf{P}$ to be the category of profinite domains and continuous functions. □

This differs from the category of profinite posets as defined in [Niño 1981] in that the profinite posets considered here need not have least elements.

Theorem 37 *The following are equivalent for any cpo D.*

1. *D is profinite.*
2. *D is algebraic and $\mathbf{B}[D]$ is a Plotkin order.*
3. *D is isomorphic to a $\mathbf{CPO}^{\uparrow}$ inverse limit of finite posets.*
4. *There is a directed set M of finite deflations on D such that $\bigsqcup M = \mathrm{id}_D$.*
5. *D is algebraic and there is a directed set M of continuous functions $f : D \to D$ such that $im(f)$ is finite and $\bigsqcup M = \mathrm{id}_D$.*

Proof. By Proposition 36, the category $\mathbf{Id}^{\uparrow}_{\mathbf{PLT}}$ has limits for inverse systems. Since all of the finite posets are objects in $\mathbf{Id}^{\uparrow}_{\mathbf{PLT}}$, it follows that (1) $\Rightarrow$ (2). That (2) $\Rightarrow$ (3) is immediate. If $\mu : D \to \Delta$ is initial in $\mathbf{CPO}^{\uparrow}$ and the posets in Δ are finite then $M = \{\mu_i^{\downarrow} \circ \mu_i \mid i \in I\}$ satisfies the conditions of (4) so (3) $\Rightarrow$ (4). The cone of projections determined by a set M satisfying (4) gives D as a $\mathbf{CPO}^P$ inverse limit of finite posets so (4) $\Rightarrow$ (1).

To complete the proof we show that (4) $\Leftrightarrow$ (5). That (4) $\Rightarrow$ (5) is immediate. Suppose $f : D \to D$ is a continuous function with a finite image such that $f(x) \sqsubseteq x$ for each x. Then for any n and any x, $f^{n+1}(x) \sqsubseteq f^n(x)$. Since f has a finite image it follows that for some m, $f^{m+1}(x) = f^m(x)$. So define $f_\infty : D \to D$ by setting $f_\infty(x) = f^m(x)$ where $f^{m+1}(x) = f^m(x)$. This function is monotone, for if $x \sqsubseteq y$, $f^m(x) = f_\infty(x)$ and $f^n(y) = f_\infty(y)$ then for any $l \geq m, n$ we have $f_\infty(x) = f^l(x) \sqsubseteq f^l(y) = f_\infty(y)$. Since the image of f_∞ is finite, it follows that f_∞ is continuous. Moreover, if $x \in D$ and $f^{n+1}(x) = f^n(x)$ then $f_\infty^2 = f^{2n}(x) = f^n(x) = f_\infty(x)$ so f_∞ is a finite deflation. The set $M_\infty = \{f_\infty \mid f \in M\}$ is directed so there is a continuous function $g = \bigsqcup M_\infty$. We claim that g is the identity map on D. To see this, suppose $e \in \mathbf{B}[D]$. Now $e = (\bigsqcup M)(e)$ so $e \sqsubseteq f(e)$ for some $f \in M$. Hence $e = f(e) = f_\infty(e)$. Thus $g(e) = e$ and since D is algebraic we conclude that g is the identity function. The conditions of (4) are therefore satisfied. □

Theorem 37 has many noteworthy consequences. The remainder of this chapter is devoted to listing some of them. The theorem is used in so many places in the remaining chapters that it it will not always be mentioned explicitly. The equivalence between (1) and (2) is used especially often. In light of Proposition 36, the following is immediate:

Corollary 38 $\mathbf{P}^{\uparrow}$ *has limits for inverse systems.* □

We now quote an observation of Bracho [1983] that the poset of algebraic deflations on a profinite poset form an algebraic lattice. Indeed, the results we have proved above allow us to see that this lattice has an even stronger property:

Corollary 39 *For any profinite poset D, the poset of algebraic deflations on D is an algebraic lattice with a locally finite basis.*

Proof. This is am immediate consequence of Propositions 21 and Theorem 37. □

Now, if A is a countable Plotkin poset then there is a chain $A_0 \triangleleft A_1 \triangleleft \cdots$ of finite normal substructures of A with $A = \bigcup_{n \in \omega} A_n$. Hence, if D is an ω-algebraic cpo (*i.e.* $\mathbf{B}[D]$ is countable) then it is the ideal completion of a Plotkin order with a trivial root if and only if it is the limit of a countable sequence of finite posets having least elements. Hence Plotkin's orginal name for the category of limits of countable $\mathbf{CPO}^P$ inverse systems of finite posets (having least elements) was **SFP** (for **S**equence of **F**inite **P**osets). Objects of this category are called *strongly algebraic domains*. This is the equivalence which is demonstrated in [Plotkin 1976]. In [Niño 1981] it is shown that conditions (1), (2) and (3) in Theorem 37 are equivalent when the posets involved are assumed to have least elements. Equivalence of conditions (4) and (5) with the other conditions came out of conversations between the author, Gordon Plotkin and Dana Scott.

Corollary 40 *Let* **BotP** *be the category of profinite domains having bottom elements and continuous functions. Then* $\mathbf{BotP}^P$ *has limits for inverse systems.* □

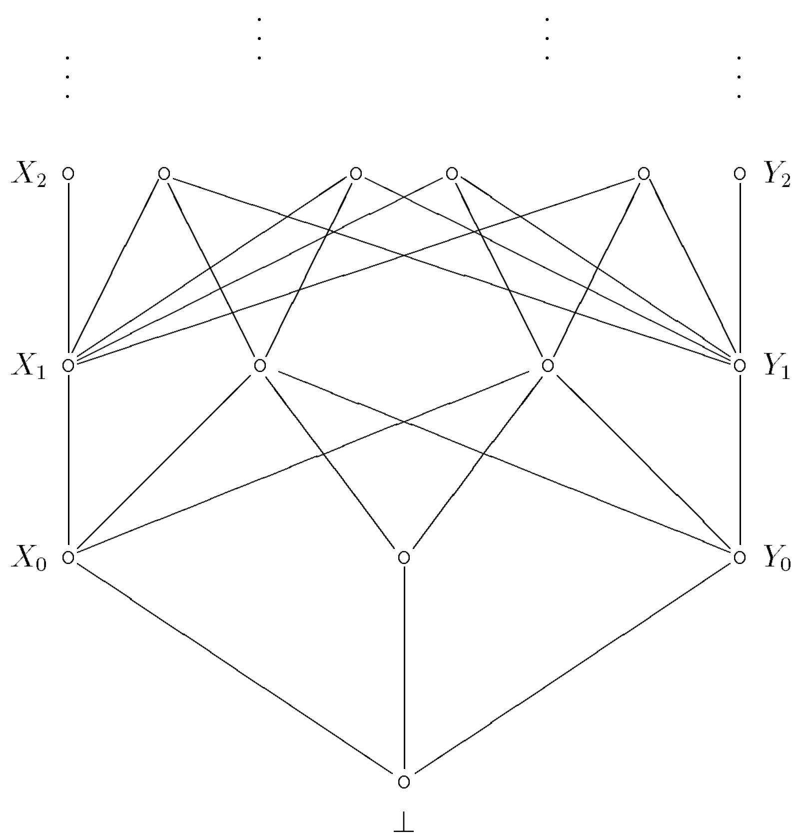


Figure 3.1: A Plotkin poset A such that $|A|$ does not have property M.

Lemma 41 *Profinite domains have greatest lower bounds for filtered systems. In other words, if D is profinite then D^{op} is a cpo.*

Proof. Suppose $\Im \subseteq D$ is filtered and let M be a directed collection of finite deflations such that $\bigsqcup M = \mathrm{id}_D$. For each $p \in M$, $p(\Im)$ is finite so there is a finite $u \subseteq D$ such that $p(u) = p(\Im)$. If $x \in \Im$ is a lower bound for u then $p(x)$ is a least element of $p(\Im)$. Hence $\min p(\Im)$ exists for each $p \in M$. Now, if $p \sqsubseteq q$ then $\min p(\Im) \sqsubseteq \min q(\Im)$ so the set $N = \{\min p(\Im) \mid p \in M\}$ is directed. Let $x = \bigsqcup N$ and suppose $y \in \Im$. Then $\min p(\Im) \sqsubseteq p(y)$ so

$$x = \bigsqcup_{p \in M} \min p(\Im) \sqsubseteq \bigsqcup_{p \in M} p(y) = y.$$

Hence $x \sqsubseteq \Im$. But if $x \sqsubseteq x' \sqsubseteq \Im$ and $p \in M$ then $\min p(\Im) \sqsubseteq p(x') \sqsubseteq p(\Im)$ so $p(x') = \min p(\Im)$. Thus

$$x' = \bigsqcup_{p \in M} p(x') = \bigsqcup_{p \in M} \min p(\Im) = x$$

and x is therefore a greatest lower bound for $\Im$. □

Corollary 42 *Any profinite domain has property m.* □

It is not the case, however, that a profinite domain always has property M. A Plotkin poset A such that $|A|$ does not have property M can be obtained by adding two chains $\langle X_n \rangle$ and $\langle Y_n \rangle$ to the binary braching tree as pictured in Figure 3.1. The ideals $\downarrow\{X_n \mid n \in \omega\}$ and $\downarrow\{Y_n \mid n \in \omega\}$ have 2^ω minimal upper bounds in $|A|$. Since Plotkin orders have property M, this example also shows that a profinite domain need not be a Plotkin order. Moreover, although $|A|$ is profinite, its ideal completion cannot be profinite because it is not a Plotkin order. Hence the class of profinite domains is *not* closed under ideal completion.

Chapter 4

Some Distinguished Categories of Cpo's

This chapter is devoted to studying a variety of miscellaneous categories which are related to the profinites. Some of these are distinguished by various relevant properties, like that of being cartesian closed or having certain interesting "definability properties". We also discuss the Scott topology on profinite domains and some other classes of posets. The last section defines a compact Hausdorff topology on profinite domains.

4.1 Extensions of Smyth's theorem

There is an interesting characterization of **SFP** which has important implications for the search for new cartesian closed categories of domains. The following theorem was proved by Smyth [1983] and answers a conjecture of Gordon Plotkin.

Theorem 43 *If D and $\mathbf{CPO}(D, D)$ are ω-algebraic cpo's with bottoms then D is profinite.* □

Smyth also shows that for any full cartesian closed sub-category of **SFP** the product and exponentation functors must be exactly the ones we have defined. This yields the following:

Theorem 44 **SFP** *is the largest cartesian closed full sub-category of ω-algebraic cpo's with bottoms.* □

Proposition 45 *If D is a cpo with a least element $\perp$ and $\mathbf{CPO}(D, D)$ is ω-algebraic, then D is ω-algebraic.*

Proof. Suppose $f : D \to D$ is finite (as an element of $\mathbf{CPO}(D, D)$). We claim that $f(\perp)$ is finite. Suppose $a_0 \sqsubseteq a_1 \sqsubseteq \cdots$ is a chain in D with $f(\perp) = \bigsqcup_n a_n$. For each n, define $f_n : D \to D$ by

$$f_n(x) = \begin{cases} f(x) & \text{if } x \neq \perp; \\ a_n & \text{if } x = \perp. \end{cases}$$

These functions are all continuous and $\bigsqcup_n f_n = f$ so $f = f_n$ for some n. Hence $f(\perp) = f_n(\perp) = a_n$ and $f(\perp)$ must therefore be finite. Now, suppose $d \in D$ and let $f(x) = d$ be the constant function determined by d. Since $\mathbf{CPO}(D, D)$ is ω-algebraic, there are finite functions $f_0 \sqsubseteq f_1 \sqsubseteq \cdots$ such that $f = \bigsqcup_n f_n$. Hence $\bigsqcup_n f_n(\perp) = f(\perp) = d$. But $f_n(\perp)$ is finite for each n so D must be algebraic. □

Corollary 46 *If D is a cpo with a least element and $\mathbf{CPO}(D, D)$ is an ω-algebraic cpo then D is profinite.* □

Theorem 47 *If D is an ω-algebraic cpo and $\mathbf{CPO}(D, D)$ is ω-algebraic then D is profinite.*

Proof. The proof is quite lengthy and is divided into three claims corresponding roughly to the three cases pictured in Figure 2.1. The first claim is the most difficult and corresponds to Figure 2.1a. The proof of that claim is offered below in some detail. Proofs of the other two claims do not differ much from those given in [Smyth 84] for similar cases (see Theorems 3 and 4 there). Let E be the poset of continuous functions from D into D.

1. If $\mathbf{B}[D]$ does not have property m then E is not ω-algebraic.

   Proof. Suppose $\mathbf{B}[D]$ fails to have property m. Then there is a finite set $u \subseteq \mathbf{B}[D]$ and a sequence $\{z_n \mid n \in \omega\}$ in $\mathbf{B}[D]$ such that

   - $z_n \sqsupset z_m$ for each $n \leq m$;
   - $z_n \sqsupset u$ for each n;
   - if $z_n \sqsupseteq x$ for each n then $x \not\sqsupseteq u$.

   Now, suppose $f : \omega \to \omega$ is monotone and $n \geq f(n)$ for each n. Define a function $\tilde{f} : \mathbf{B}[D] \to D$ as follows.

$$f(x) = \begin{cases} x & \text{if } x \sqsubseteq z_n \text{ for each n;} \\ z_{f(n)} & \text{where } z_n \text{ is the least } z_k \text{ above } x \text{ if there is one;} \\ z_0 & \text{otherwise.} \end{cases}$$

   We show that $\tilde{f}$ is monotone. Suppose $x \sqsubseteq y$ for $x, y \in \mathbf{B}[D]$. Suppose first that $x \sqsubseteq z_n$ for each n. If y also has this property then $\tilde{f}(x) = x \sqsubseteq y = \tilde{f}(y)$. In either of the other two cases, $\tilde{f}(y) = z_m$ for some m so $\tilde{f}(x) = x \sqsubseteq \tilde{f}(y)$. Suppose n is the largest k such that $z_k \sqsupseteq x$. If there is a largest m such that $z_m \sqsupseteq y$ then we must have $m \sqsupseteq n$ so $\tilde{f}(x) = z_n \sqsubseteq z_m = \tilde{f}(y)$. If there is no such m then $\tilde{f}(x) = z_n \sqsubseteq z_0 = \tilde{f}(y)$. Finally, if there is no n such that $x \sqsubseteq z_n$ then this is also true of y so $\tilde{f}(x) = z_0 = \tilde{f}(y)$.

   Suppose $i = \widetilde{\mathrm{id}}_\omega$ where id_ω is the identity function on ω. Suppose

$$g : \mathbf{B}[D] \to D$$

   is a monotone function below i and $g(x) = x$ for each $x \in u$. If $x \sqsupseteq u$ then $g(x) \sqsupseteq u$ so either $g(x) \not\sqsupseteq z_0$ or there is a largest n such that $g(x) \sqsubseteq z_n$ (by our assumptions on the z_n's). Since $g \sqsubseteq i$, it follows that for each n there is a largest k such that $g(z_n) \sqsubseteq z_k$. Let $f : \omega \to \omega$ be the function thus determined by g. It is clearly monotone and $n \geq f(n)$

for each n. We claim that $g \sqsupseteq \tilde{f}$. If $x \sqsubseteq z_n$ for each n then $g(x) \sqsubseteq i(x) = x = \tilde{f}(x)$. If $x \sqsupseteq u$ then $\tilde{f}(x)$ is the least $z_n \sqsupseteq g(x)$ (and there always is such a z_n because $i \sqsupseteq g$).

Now, consider the functions $f_n : \omega \to \omega$ by

$$f_n(x) = \begin{cases} f(k) & \text{if } k \leq n; \\ f(k) + 1 & \text{otherwise.} \end{cases}$$

Note that $\bigsqcup f_n = f$ so $\bigsqcup \tilde{f}_n = \tilde{f}$. But for no n is it possible that $g \sqsubseteq \tilde{f}_n$. For if this were the case then for $k \sqsupseteq n$ we would have $g(z_k) \sqsubseteq \tilde{f}_n(z_k) = z_{f(k)+1}$ which contradicts the definition of f. This shows that no $g \sqsubseteq i$ which fixes u can be finite in the poset of monotone functions from $\mathbf{B}[D]$ into D.

Suppose $g_0 \sqsubseteq g_1 \sqsubseteq \cdots$ is a sequence of monotone functions from $\mathbf{B}[D]$ into D such that $\bigsqcup g_n = i$. Then there is an n such that $g_n(x) = x$ for each $x \in u$. But we have just shown that no such g can be finite. It follows that the poset of monotone functions form $\mathbf{B}[D]$ into D cannot be ω-algebraic. But by Lemma 4 this poset is isomorphic to E. This proves the claim. □

2. If $\mathbf{B}[D]$ has property m but does not have property M then $\mathbf{B}[E]$ has continuum many finite elements and therefore cannot be ω-algebraic. (This case corresponds to Figure 2.1b.) □

3. If $\mathbf{B}[D]$ has property M but is not a Plotkin order then E is not ω-algebraic. (This case corresponds to Figure 2.1c.) □

Since the three claims exhaust all of the cases in which the basis of D is not a Plotkin order, it follows that D must be profinte. □

4.2 Bounded complete cpo's

Definition: A pre-order $\langle A, \vdash \rangle$ is said to be *bounded complete* if A is non-empty and for every bounded finite $u \subseteq A$ there is an $X \in A$ such that $X \vdash u$ and if $Y \vdash u$ then $Y \vdash X$. Such an element X is called a least upper bound for u. □

Lemma 48 *Let* **BCALG** *be the category of bounded complete algebraic cpo's and continuous functions.*

1. $\mathbf{BCALG} = \mathbf{Id}_{\mathcal{K}}$ *where $\mathcal{K}$ is the class of bounded complete pre-orders.*

2. **BCALG** *is cartesian closed.*

3. $\mathbf{BCALG} \subseteq \mathbf{P}$.

4. **BCALG** *is closed under homomorphisms.*

5. $\mathbf{BCALG}^{\uparrow}$ *has inverse limits.*

Proof. We omit the well-known proofs of (1) and (2). If A is bounded complete then it surely has property m. If $u \subseteq A$ is finite then $\mathcal{U}^1(u)$ is finite but one can show that $\mathcal{U}^1(u) = \mathcal{U}^*(u)$. Hence A is a Plotkin order and $|A|$ is therefore profinite. Hence (3) follows from (1). To prove (4) and (5) one shows that $\mathcal{K}$ is closed under normal substructures and normal directed unions. □

As an immediate corollary of the lemma, note that all upper semi-lattices are Plotkin posets. The bounded complete posets form a particulary nice class in a number of regards. One of these concerns the notion of *first order axiomatizability.* All of the necessary facts and definitions from first order model theory can be found in [Barwise 1977]. We will need the following:

Lemma 49 *Let A and B be posets and suppose $u \subseteq A \times B$. If $X \in A$ and $Y \in B$ then* $\mathrm{MUB}(u) = \mathrm{MUB}(\mathrm{fst}(u)) \times \mathrm{MUB}(\mathrm{snd}(u))$ *where*

$$\mathrm{fst}(u) = \{X' \in A \mid (X', Y') \in u \textit{ for some } Y' \in B\}, \textit{ and}$$
$$\mathrm{snd}(u) = \{Y' \in B \mid (X', Y') \in u \textit{ for some } X' \in A\}.$$

□

Proof. First suppose $(X, Y) \in \mathrm{MUB}(u)$. Then clearly $X \sqsupseteq \mathrm{fst}(u)$. If $X \sqsupseteq X' \sqsupseteq \mathrm{fst}(u)$ then $(X, Y) \sqsupseteq (X', Y) \sqsupseteq u$ so $X = X'$. Hence $X \in \mathrm{MUB}(\mathrm{fst}(u))$. A similar argument shows that $Y \in \mathrm{MUB}(\mathrm{snd}(u))$. To prove the converse, suppose $X \in \mathrm{MUB}(\mathrm{fst}(u))$ and $Y \in \mathrm{MUB}(\mathrm{snd}(u))$. Clearly, $(X, Y) \sqsupseteq u$. If $(X, Y) \sqsupseteq (X', Y') \sqsupseteq u$ then $X \sqsupseteq X' \sqsupseteq \mathrm{fst}(u)$ and $Y \sqsupseteq Y' \sqsupseteq \mathrm{snd}(u)$ so $(X, Y) = (X', Y')$. Thus $(X, Y) \in \mathrm{MUB}(u)$. □

Definition: Let us say that a class of models $\mathcal{K}$ for a first order language $\mathcal{L}$ is a *countably Δ-elementary class* if there is a set T of first order sentences in an expansion of $\mathcal{L}$ such that the class of reducts to $\mathcal{L}$ of countable models of T is $\mathcal{K}$. □

Proposition 50 *The class of bounded complete posets is the largest countably Δ-elementary class of posets having property* M *which is closed under the product operation.*

Proof. Let T be a first order theory for a language $\mathcal{L}$ having a binary relation symbol $\preceq$ and suppose T contains the poset axioms for $\preceq$. Suppose, moreover, that if A is a model of T then $A \times A$ is a model of T and that every model of T has property M. Let A be a model of T in which the interpretation of $\preceq$ is not bounded complete. Then there is a finite (possibly empty) set $u \subseteq A$ such that $\mathrm{MUB}(u)$ has at least two elements. Suppose u has n elements. For each integer $m \geq 2$ we show that there is a model of T satisfying the axiom

$$\phi_m \equiv \exists \mathbf{v}_1 \cdots \exists \mathbf{v}_m [(\bigwedge_{i \neq j} \mathbf{v}_i \neq \mathbf{v}_j) \wedge (\mathbf{v}_i \in \mathrm{MUB}(\{\mathbf{c}_1, \ldots, \mathbf{c}_m\}))]$$

for constants $\mathbf{c}_1, \ldots, \mathbf{c}_n$ not contained in $\mathcal{L}$. It is easy to check that ϕ_m really is a first order statement. Note that A is a model of ϕ_2 if $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are interpreted by the elements of u. So suppose we know that $T \cup \{\phi_m\}$ has a model B in which $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are interpreted by $X_1, \ldots, X_n$. We claim that $B \times B$ is a model of $T \cup \{\phi_{m+1}\}$ when $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are interpreted by

$(X_1, X_1), \ldots, (X_n, X_n)$. To see this, let $v = \{X_1, \ldots, X_n\}$ and $w = \{(X_1, X_1), \ldots, (X_n, X_n)\}$. Then

$$\begin{aligned} \mathrm{MUB}(w) &= \mathrm{MUB}(\mathrm{fst}(w)) \times \mathrm{MUB}(\mathrm{snd}(w)) && \text{by 49} \\ &= \mathrm{MUB}(v) \times \mathrm{MUB}(v). \end{aligned}$$

Since $m > 1$ there are m^2 elements in $\mathrm{MUB}(v) \times \mathrm{MUB}(v)$. Since $m^2 > m + 1$ for $m > 1$ we are done. Now, for each m, $\phi_{m+1} \rightarrow \phi_m$ so we may deduce that any finite subset of $T \cup \{\phi_m \mid m \geq 2\}$ has a model. Hence, by the Compactness Theorem, there is a model C of $T \cup \{\phi_m \mid m \geq 2\}$. But if C interprets $\mathbf{c}_1, \ldots, \mathbf{c}_n$ by $\{Y_1, \ldots, Y_n\}$ then $\mathrm{MUB}(\{Y_1, \ldots, Y_n\})$ must be infinite. Hence C cannot have property M, contradicting the assumption on models of T. We conclude that all of the models of T must be bounded complete. □

Corollary 51 *If $\mathcal{K}$ is a countably Δ-elementary class of posets and $\mathbf{Id}_{\mathcal{K}}$ is cartesian closed then $\mathbf{Id}_{\mathcal{K}} \subseteq \mathbf{BCALG}$.*

Proof. Immediate from Theorem 47 and Proposition 50. □

The category of bounded complete algebraic cpo's and continuous functions has two interesting sub-categories which we mention later. A poset A is *coherent* when for every finite $u \subseteq A$, if u is pairwise bounded then u has a least upper bound. It is not hard to check that the algebraic cpo's having coherent bases form a cartesian closed full sub-category of the conisistently complete algebraic cpo's. However, the best known sub-category is that of algebraic lattices. One can show that these form a ccc and are exactly the algebraic cpo's with a basis that is an upper semi-lattice. The methods used in the proof of Lemma 48 can be used to show that these two categories also have the properties (2)–(5) listed there.

An *information system* is a 4-tuple

$$\langle \mathcal{D}, \Delta_{\mathcal{D}}, \mathrm{Con}_{\mathcal{D}}, \vdash_{\mathcal{D}} \rangle$$

where $\Delta_{\mathcal{D}} \in D$, $\mathrm{Con}_{\mathcal{D}}$ is a set of finite subsets of D and $\vdash_{\mathcal{D}}$ is a binary relation between $\mathrm{Con}_{\mathcal{D}}$ and $\mathcal{D}$. They must satisfy the following axioms:

1. $u \in \mathrm{Con}_{\mathcal{D}}$, whenever $u \subseteq v \in \mathrm{Con}_{\mathcal{D}}$,
2. $\{X\} \in \mathrm{Con}_{\mathcal{D}}$, whenever $X \in \mathcal{D}$,
3. $u \cup \{X\} \in \mathrm{Con}_{\mathcal{D}}$, whenever $u \vdash_{\mathcal{D}} X$,
4. $u \vdash_{\mathcal{D}} \Delta_{\mathcal{D}}$ for every $u \in \mathrm{Con}_{\mathcal{D}}$,
5. $u \vdash_{\mathcal{D}} X$, whenever $X \in u$,
6. if $v \vdash_{\mathcal{D}} Y$ for all $Y \in u$ and $u \vdash_{\mathcal{D}} X$, then $v \vdash_{\mathcal{D}} X$.

An arrow $f : \mathcal{D} \rightarrow \mathcal{E}$ between information systems $\mathcal{D}$ and $\mathcal{E}$ is a relation between $\mathrm{Con}_{\mathcal{D}}$ and $\mathrm{Con}_{\mathcal{E}}$ such that:

1. $\emptyset \; f \; \emptyset$,

2. if $u\ f\ v$ and $u\ f\ w$ then $u\ f\ (v \cup w)$,

3. if $u' \vdash_{\mathcal{D}} X$ for each $X \in u$, $u\ f\ v$ and $v \vdash_{\mathcal{D}} Y$ then $u'\ f\ \{Y\}$.

If $g : \mathcal{D} \to \mathcal{E}$ and $f : \mathcal{E} \to \mathcal{F}$ are arrows between information systems $\mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ then $f \circ g : \mathcal{D} \to \mathcal{F}$ is given by letting $u\ (f \circ g)\ v$ if and only if there is a v such that $u\ g\ v$ and $v\ f\ w$. The arrow $\mathrm{id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ given by letting $u\ \mathrm{id}_{\mathcal{D}}\ v$ if and only if $u \vdash_{\mathcal{D}} Y$ for each $Y \in v$ is a two-sided identity for the $\circ$ operation. It is straight-forward to verify that the information systems with these arrows form a category which we shall call **ISYS**.

There is a close relationship between the category of bounded complete pre-orders with approximable relations and **ISYS**. We show that in a very direct way, these categories are equivalent. Let an information system $\mathcal{D}$ be given and let $\vdash$ be the binary relation on $\mathrm{Con}_{\mathcal{D}}$ given by $u \vdash v$ if and only if $u \vdash_{\mathcal{D}} X$ for each $X \in v$. That $\vdash$ is transitive and reflexive follows from axioms (5) and (6). Axiom (4) asserts that $\mathrm{Con}_{\mathcal{D}}$ has a least element with respect to $\vdash$. Suppose $u, v, w \in \mathrm{Con}_{\mathcal{D}}$ and $w \vdash u, v$. Repeated application of (3) shows that $u \cup v \cup w$ is consistent so $u \cup v$ is consistent by (6). Hence $\mathrm{Con}_{\mathcal{D}}$ is a bounded complete pre-order. The unused axiom (2) is a non-triviality assumption which prevents $\mathcal{D}$ from having superfluous members.

If $f : \mathcal{D} \to \mathcal{E}$ is an arrow between information systems then $f : \mathrm{Con}_{\mathcal{D}} \to \mathrm{Con}_{\mathcal{E}}$ is an approximable relation and the composition for approximable relations is identical to that for information systems. To complete the proof of equivalence we must show how to obtain an information system from a bounded complete pre-order and show that this operation is inverse to the one given above for getting a pre-order from an information system. So let A be a bounded complete pre-order and suppose Con is the set of finite bounded subsets of A. If $u, v \in \mathrm{Con}$ then say $u \vdash v$ if and only if $X \vdash_A u$ implies $Y \vdash_A v$. If Δ is any least upper bound of $\emptyset$ it is easy to check that $\langle A, \Delta, \mathrm{Con}, \vdash \rangle$ is an information system. Suppose $\vdash_0$ is the ordering induced by $\vdash$ on Con. Say

$$f : \langle A, \vdash_A \rangle \to \langle \mathrm{Con}, \vdash_0 \rangle$$

is given by $X\ f\ u$ if an only if $X \vdash_A u$. Let

$$g : \langle \mathrm{Con}, \vdash_0 \rangle \to \langle A, \vdash_A \rangle$$

be given by $u\ g\ X$ if an only if $X \vdash_A u$ implies $Y \vdash_A X$. That f is approximable is immediate from its definition. To see that g is approximable, suppose $u\ g\ X$ and $u\ g\ Y$. Since A is bounded complete, there is a least upper bound Z for $\{X, Y\}$. If $Z' \vdash_A u$ then $Z' \vdash X, Y$ so $Z' \vdash_A Z$. Hence $u\ g\ Z'$. The other conditions for approximability of g are easy to check. That f and g are inverses of each other is also easy to see. Thus $\langle A, \vdash_A \rangle \cong \langle \mathrm{Con}, \vdash_0 \rangle$ and we may conclude that **ISYS** and the pre-orders with approximable relations are equivalent categories.

4.3 Other cartesian closed sub-categories

To obtain a cartesian closed category of algebraic cpo's which is closed under $\mathbf{CPO}^P$ inverse limits one can begin with any class $\mathcal{K}_0$ of finite posets that includes 1 and proceed as follows.

First, close the class under product and exponential operations to obtain a class $\mathcal{K}_1$. Now, let $\mathcal{K}$ be the class of cpo's isomorphic to limits of $\mathbf{CPO}^P$ inverse systems from $\mathcal{K}_1$. The resulting class $\mathcal{K}$ will have the desired closure properties. The reason this works involves facts about algebroidal categories (see [Smyth 1978]) and the continuity of functors (which will be discussed later).

There may be something unsatisfactory about the result one obtains in this way, however. Unless a profinite domain D is given as an inverse limit of posets from $\mathcal{K}_1$ it may be quite difficult to tell whether D is in $\mathcal{K}$. Indeed, it may be difficult to tell whether D is in $\mathcal{K}_1$! What we are facing is the *intrinsic characterization problem.* Apparently the question, "is D in $\mathcal{K}$?" must be answered by locating an appropriate inverse system in $\mathcal{K}_1$. Unless the order structure of D provides some hint as to the proper choice of inverse system we have little hope of answering the question. The trick to understanding $\mathcal{K}$, therefore, is to provide some simple order property of posets which, if satisfied by D, qualifies it for membership in $\mathcal{K}$. In other words, it is desirable to characterize $\mathcal{K}$ intrinsically.

We have seen two good examples of this. If $\mathcal{K}_0$ is the class of all finite posets then a poset D is in $\mathcal{K}$ if and only if it is an algebraic cpo and $\mathbf{B}[D]$ is a Plotkin order. As a second example, if $\mathcal{K}_0$ is the class of finite bounded complete posets then $\mathcal{K}$ is the class of bounded complete algebraic cpo's. These two examples have several things in common. In particular, both classes $\mathcal{K}_0$ are closed under normal substructures and in both cases $\mathcal{K}_0 = \mathcal{K}_1$. What we illustrate below is that when a class of finite posets satisfies these two conditions then it will generate a very pleasant category of profinite domains. This, in essence, transforms an infinitary closure problem into a finitary one. The illustration is by way of example; we present a new cartesian closed category of what are here called short posets.

Definition: Suppose A is a pre-order with property M. Then A is *short* if for every finite non-empty $u \subseteq A$ and pair $X, Y \in \mathrm{MUB}(u)$, either $\{X, Y\}$ is unbounded or $X \sim Y$ (*i.e.* $X \vdash Y$ and $Y \vdash X$). □

Proposition 52 *A pre-order A with property M is short if and only if for every finite subset $u \subseteq A$, $\mathcal{U}^1(u) = \mathcal{U}^*(u)$.*

Proof. We prove necessity ($\Rightarrow$) by contradiction. Suppose $u \subseteq A$ is finite and $\mathcal{U}^1(u) \neq \mathcal{U}^2(u)$. Say $Z \in \mathcal{U}^2(u) - \mathcal{U}^1(u)$ and suppose $Z \sqsupseteq Z' \in \mathrm{MUB}(u \cup {\downarrow}Z)$. We cannot have $Z \in \mathrm{MUB}({\downarrow}Z')$ so there is some $Y \in \mathcal{U}^1(u) - {\downarrow}Z'$ such that $Z \sqsupseteq Y$. Note that $Y \in \mathrm{MUB}(u \cup {\downarrow}Y)$. Say $Z' \sqsupseteq X \in \mathrm{MUB}(u \cup {\downarrow}Y)$. If $X \sqsubseteq Y$ or $Y \sqsupseteq X$ then $Y = X \sqsubseteq Z'$ which is contrary to our assumption on Y. Hence X, Y are distinct elements of $\mathrm{MUB}(u \cup {\downarrow}Y)$ bounded by Z. Thus A cannot be short. To see the other direction ($\Leftarrow$) suppose A is not short. Then there is a finite $u \subseteq A$ and a distinct pair $X, Y \in \mathrm{MUB}(u)$ such that $\{X, Y\}$ is bounded. Say $Z \in \mathrm{MUB}(\{X, Y\})$. Then $Z \notin \mathrm{MUB}(u)$ so $Z \notin \mathcal{U}^1(u)$. But $Z \in \mathcal{U}^2(u)$ so $\mathcal{U}^1(u) \neq \mathcal{U}^*(u)$. □

The following Lemma is analogous to Lemma 49.

Lemma 53 *Suppose A and B are posets and p is a finite set of functions mapping A into B. Consider the following conditions for a function $f : A \to B$.*

1. *f is monotone minimal upper bound for p.*

2. *for each $X \in A$, $f(X)$ is a minimal upper bound for*

$$\{g(X) \mid g \in p\} \cup \{f(Y) \mid Y \sqsubset X\}$$

Then (1) $\Rightarrow$ (2). If A is well-founded (i.e. every subset of A has a minimal element) then (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2). Suppose $f \in \mathrm{MUB}(p)$ is monotone and there is an X such that $f(X) \notin \mathrm{MUB}(t)$ where

$$t = \{g(X) \mid g \in p\} \cup \{f(Y) \mid Y \sqsubset X\}. \tag{$*$}$$

Say $Z \in B$ such that $t \sqsubseteq Z \sqsubset f(X)$. Define $f' : A \to B$ by

$$f'(Y) = \begin{cases} f(Y) & \text{if } Y \neq X; \\ Z & \text{if } Y = X. \end{cases}$$

Since $Z \sqsubset f(X)$ we know that $f' \sqsubset f$. Suppose $g \in p$. If $Y \neq X$ then $g(Y) \sqsubseteq f(Y) = f'(Y)$. But $g(X) \in t$ so $g(X) \sqsubseteq Z = f'(X)$. Hence $p \sqsubseteq f'$. To see that f' is monotone, suppose $Y, Y' \in A$ and $Y \sqsubseteq Y'$. We treat three separate cases. Case 1. If $Y, Y' \neq X$ then $f'(Y) = f(Y) \sqsubseteq f(Y') = f'(Y')$. Case 2. If $X \sqsubset Y$ then $f'(X) = Z \sqsubseteq f(X) \sqsubseteq f(Y) = f'(Y')$. Case 3. If $Y \sqsubset X$ then $f'(Y) = f(Y) \in t$ and $t \sqsubseteq Z = f'(X)$ so $f'(Y) \sqsubseteq f'(X)$. This shows that f' is monotone so $f \notin \mathrm{MUB}(p)$ a contradiction.

(2) $\Rightarrow$ (1). Suppose A is well-founded. Let $f' : A \to B$ be a monotone function such that $f \sqsupseteq f' \sqsupseteq p$. If $E = \{Z \mid f(Z) \neq f'(Z)\} \neq \emptyset$ then there is a minimal element X of E. Let t be defined as in $(*)$. Then

$$t = \{g(X) \mid g \in p\} \cup \{f'(Y) \mid Y \sqsubset X\}$$

because of the minimality of X. Hence $f(X) \sqsupseteq f'(X) \sqsupseteq t$. But $f(X) \in \mathrm{MUB}(t)$ so $f(X) = f'(X)$ which contradicts the choice of X. Hence $E = \emptyset$ and $f = f'$. Moreover, f is monotone because $f(X) \sqsupseteq \{f(Y) \mid Y \sqsubset X\}$ for each X. □

Lemma 54 *If A and B are finite, short posets then 1, $A \times B$ and* $\mathbf{CPO}(A, B)$ *are short.*

Proof. It is obvious that 1 is short. Let $p \subseteq A \times B$ and suppose (X, Y) and (X', Y') are distinct minimal upper bounds of p. Then either $X \neq X'$ or $Y \neq Y'$. If $X \neq X'$ then $\{X, X'\}$ is unbounded since X and X' are minimal upper bounds of $\mathrm{fst}(p)$ (by Lemma 49) and A is short. Thus $\{(X, Y), (X', Y')\}$ is unbounded. The other case ($Y \neq Y'$) is essentially the same. We conclude that $A \times B$ is short.

Let $p \subseteq \mathbf{CPO}(A, B)$ and suppose $f, g \in \mathbf{CPO}(A, B)$ are minimal upper bounds for p. Since A is finite, there is a minimal $X \in A$ such that $f(X) \neq g(X)$. By Lemma 53, $f(X)$ and $g(X)$ are minimal upper bounds of

$$s = \{h(X) \mid h \in p\} \cup \{f(Y) \mid Y \sqsubset X\}, \text{ and}$$
$$t = \{h(X) \mid h \in p\} \cup \{g(Y) \mid Y \sqsubset X\}$$

respectively. But $s = t$ since $f(Y) = g(Y)$ for each $Y \sqsubset X$. Since B is short, it follows that $\{f(X), g(X)\}$ is unbounded. Hence $\{f, g\}$ is unbounded and $\mathbf{CPO}(A, B)$ is short. □

Corollary 55 *The category of short pre-orders and approximable relations is cartesian closed. Hence the equivalent category of algebraic cpo's with short bases and continuous maps is cartesian closed. Moreover, this latter category is closed under* $\mathbf{CPO}^{\uparrow}$ *inverse limits.*

Proof. Suppose A and B are short and consider the pre-order B^A. If $u \subseteq B^A$ is finite then there are finite normal substructures $M \lhd A$ and $N \lhd B$ such that $u \subseteq N^M \lhd B^A$. Since A and B are short, M and N are also short. But N^M is isomorphic to $\mathbf{CPO}(\tilde{M}, \tilde{N})$ which is short by Lemma 54. From this it follows that B^A is short. A similar proof shows that $A \times B$ is short if A and B are short. Hence the short pre-orders form a ccc. It is easy to show that the short pre-orders are closed under normal substructures and normal directed unions so the algebraic cpo's with short bases have $\mathbf{CPO}^{\uparrow}$ inverse limits by Corollary 35. □

4.4 The Scott topology

Definition: Let D be a cpo. If $x, y \in D$ then we say that y is *way below* x and write $y \ll x$ if for every directed subset $M \subseteq D$, $\bigsqcup M \sqsupseteq x$ implies $z \sqsupseteq y$ for some $z \in M$. D is *continuous* if there is a set $B \subseteq D$ called a *basis* for D such that for each $x \in D$, the set $\hat{x} = \{y \in B \mid x \gg y\}$ is directed and $x = \bigsqcup \hat{x}$. □

Continuous *lattices* were introduced by Dana Scott [1972] as a generalization of algebraic lattices. The theory of continuous lattices is given a detailed treatment in [Gierz *et. al.* 1980]. A leisurely discussion of continuous cpo's with least elements appears in [Weihrauch and Deil 1980]. Note that we have not assumed that a continuous cpo has a least element. One can show that the continuous lattices are exactly the continuous retracts of algebraic lattices. Much of the more general theory of continuous cpo's can be developed by analogy with that of continuous lattices. For example, the continuous cpo's are exactly the continuous retracts of algebraic cpo's. In fact we have the following

Proposition 56 *If D is a continuous cpo with basis B then D is a continuous projection of $|B|$.*

Proof. Let D be a continuous cpo with basis B. Let $p : |B| \to D$ by $p(M) = \bigsqcup M$ for any ideal $M \subseteq B$ and let $q : D \to |B|$ by $q(x) = \{y \in B \mid x \gg y\}$. It is obvious that p is continuous and q is monotone. Now, for $x \in D$,

$$\begin{aligned} p \circ q(x) &= p(\{y \in B \mid x \gg y\}) \\ &= \bigsqcup\{y \in B \mid x \gg y\} \quad . \\ &= x \end{aligned}$$

To complete the proof we need the following fact from the theory of continuous cpo's: if $\bigsqcup M \gg y$ for a directed set M then $x \gg y$ for some $x \in M$ (see Lemma 3.3 in [*op. cit.*]). Thus, for $M \subseteq B$ an ideal,

$$\begin{aligned} q \circ p(M) &= q(\bigsqcup M) \\ &= \{y \mid \bigsqcup M \gg y\} \\ &\subseteq M \end{aligned}$$

so $\langle p, q\rangle : |B| \xrightarrow{\text{pe}} D$ is continuous. □

We will use assume the following basic fact from the topological theory of continuous cpo's. The proof is straight-forward:

Lemma 57 *If D is continuous with a basis B, then sets of the form $\{y \mid y \gg x\}$ where $y \in B$ form a basis for ΣD.* □

We also note that a continuous cpo D is algebraic if and only if it has $\mathbf{B}[D]$ (= finite elements of D) as a basis. Hence there is a close link between (topological) bases for ΣD and (order-theoretic) bases for D.

Lemma 58 *Let D be a continuous cpo. Then an open set $K \subseteq D$ is compact if and only if $K = \uparrow u$ for a finite set $u \subseteq \mathbf{B}[D]$.*

Proof. To prove necessity ($\Rightarrow$), let $x \in K$ and suppose L is a maximal descending chain in $K \cap \downarrow x$. We claim that L has a minimal element. Suppose it does not and let $C = \bigcap\{\downarrow y \mid y \in L\}$. Suppose $z \in K \cap C$. If $z \in L$ then it is a minimal element of L. But if $z \sqsubset y$ for each $y \in L$ then the maximality of L is contradicted. Hence $K \cap C = \emptyset$ and

$$K \subseteq D - C = \bigcup\{D - \downarrow y \mid y \in L\}.$$

Since K is compact, there is a finite set $u \subseteq L$ such tht $K \subseteq \bigcup\{D - \downarrow y \mid y \in u\}$. So $K \subseteq D - \downarrow y$ where y is the least element in u. But this is impossible because $y \in K$. We conclude, therefore, that L has a least element x'. Since D' is continuous, $x' = \bigsqcup\{y \in D \mid x' \gg y\}$. Since K is open, this means there is a $y \in K$ such that $x' \gg y$. But x' is minimal in K so $x' = y$. Hence $x' \gg x'$ and x' is therefore finite. We have shown that for each $x \in K$ there is a finite $x' \sqsubseteq x$ such that x' is minimal in K. Hence $K = \uparrow S$ where S is the set of minimal elements of K. Moreover, each minimal element of K is finite. Since K is compact and $\uparrow x$ is open for a finite x, there is a finite subset $u \subseteq S$ such that $K = \uparrow u$. (Indeed, S itself is finite.) The proof of the converse ($\Leftarrow$) is trivial. □

Remark: A similar proof can be used to show that an arbitrary cpo D is compact if an only if the empty set has a finite complete set of minimal upper bounds in D.

Definition: Suppose S is a set and τ is a topology on S. A subset $K \subseteq S$ is *1-Lindelöf* if whenever $\mathcal{O} \subseteq \tau$ covers K, there is an open set $O \in \mathcal{O}$ such that $K \subseteq O$. □

Corollary 59 *If D is an arbitrary cpo then $K \subseteq D$ is a 1-Lindelöf open set if and only if $K = \uparrow x$ for some $x \in \mathbf{B}[D]$.*

Proof. If K is 1-Lindelöf then by Lemma 58, $K = \uparrow u$ for a finite $u \subseteq \mathbf{B}[D]$. But K is 1-Lindelöf and $\{\uparrow y \mid y \in u\}$ is an open cover so $K = \uparrow x$ for some $x \in u$. On the other hand, if $x \in \mathbf{B}[D]$ then any open cover of $\uparrow x$ is covered by any member of the cover that contains x. □

Lemma 60 *Let A be a pre-order and suppose $S, T \subseteq A$. Then T is a complete set of minimal upper bounds for S if and only if*

$$\bigcap_{X \in S} \uparrow X = \bigcup_{Y \in T} \uparrow Y.$$

Proof. To shorten the notation, let $L = \bigcap_{X \in S} \uparrow X$ $M = \bigcup_{Y \in T} \uparrow Y$. To prove ($\Rightarrow$), suppose T is a complete set of minimal upper bounds for S and $Z \in L$. Then $X \vdash S$ so $X \vdash Y$ for some $Y \in T$. Hence $Z \in M$. On the other hand, if $Z \in M$ then $Z \vdash S$ so $Z \in L$. To prove the converse ($\Leftarrow$), suppose $L = M$ and $Y \in T$. Then $Y \in M = L$ so $Y \vdash S$. Hence T is a set of upper bounds of S. If $Z \vdash S$ then $Z \in L = M$ so $Z \vdash Y$ for some $Y \in T$. Thus T is a complete set of upper bounds. □

Definition: Let S be an set and suppose $\Im$ is a collection of subsets of S. Let us say that $\Im$ is *quasi-conjunctive* if for every finite set u of elements of $\Im$ there is a finite set v of elements of $\Im$ such that $\bigcap v = \bigcup v$. □

Theorem 61 *Let D be a continuous cpo. Let $\mathcal{B}$ be the set of compact open subsets of D and let $\mathcal{B}_1$ be the 1-Lindelöf open subsets of D. Then*

1. *The following conditions are equivalent:*
   - *(a) $\mathcal{B}$ is a basis for ΣD;*
   - *(b) $\mathcal{B}_1$ is a basis for ΣD;*
   - *(c) D is algebraic.*
2. *The following conditions are equivalent:*
   - *(a) $\mathcal{B}$ is a basis which is closed under finite intersections;*
   - *(b) $\mathcal{B}_1$ is a quasi-conjunctive basis;*
   - *(c) D is algebraic and $\mathbf{B}[D]$ has property M.*
3. *The following conditions are equivalent:*
   - *(a) $\mathcal{B}_1$ is a basis and every finite $\Im \subseteq \mathcal{B}_1$ is contained in a finite quasi-conjunctive collection $\Im' \subseteq \mathcal{B}_1$.*
   - *(b) D is profinite.*
4. *The following conditions are equivalent:*
   - *(a) $\mathcal{B}_1$ is a basis which is closed under finite intersections;*
   - *(b) D is an algebraic lattice.*

Proof. (1) It is clear from Lemma 58 and Corollary 59 that (1a) and (1b) are equivalent. To see that (1b) $\Rightarrow$ (1c), suppose $\mathcal{B}_1$ is a basis for ΣD. Suppose $x \in D$ and let $M = \mathbf{B}[D] \cap \downarrow x$. We claim that M is directed. Suppose $y, z \in M$. Then $\uparrow y$ and $\uparrow z$ are in $\mathcal{B}_1$ and $x \in \uparrow y \cap \uparrow z$. Since $\mathcal{B}_1$ is a basis for ΣD, there is a $U \in \mathcal{B}_1$ such that $x \in U \subseteq \uparrow y \cap \uparrow z$. Now $U = \uparrow x'$ for some $x' \in \mathbf{B}[D]$ so $x \sqsupseteq x' \sqsupseteq y, z$ and the claim is established. Since D is continuous, $x = \bigsqcup N$ where $N = \{y \mid x \gg y\}$. If $y \in N$ then $x \in O - \{z \mid z \gg y\}$ and O is open. Since $\mathcal{B}_1$ is a basis for ΣD, there is a U in $\mathcal{B}_1$ such that $x \in U \subseteq O$ and $U = \uparrow y'$ for some $y' \in \mathbf{B}[D]$. Hence $y \sqsubseteq y' \sqsubseteq x$ and since y was arbitrary we must have $x = \bigsqcup M \sqsubseteq \bigsqcup N$. But $\bigsqcup N \sqsubseteq x$ since $N \subseteq M$. Hence $x = \bigsqcup N$ and D is therefore algebraic. That (1c) $\Rightarrow$ (2b) is immediate from Lemma 57.

(2) Suppose (2a) holds. A finite subset of $\mathcal{B}_1$ has the form $U = \{\uparrow x \mid x \in u\}$ where u is a finite subset of $\mathbf{B}[D]$. However, each of the sets $\uparrow x$ is in $\mathcal{B}$ so $\bigcap U$ is in $\mathcal{B}$. Hence there is a finite set $v \subseteq \mathbf{B}[D]$ such that $\bigcap U = \bigcup\{\uparrow x \mid x \in v\}$. This shows that $\mathcal{B}_1$ is quasi-conjunctive. Thus (2a) $\Rightarrow$ (2b). Now, suppose $\mathcal{B}_1$ is quasi-conjunctive. By part (1) above, D is algebraic. Suppose $u \subseteq \mathbf{B}[D]$ is finite. Then $\bigcap\{\uparrow x \mid x \in u\} = \bigcup\{\uparrow y \mid y \in v\}$ for some finite $v \subseteq \mathbf{B}[D]$ since $\mathcal{B}_1$ is quasi-conjunctive. Thus, by Lemma 60, v is a complete set of upper bounds for u. We conclude that (2b) $\Rightarrow$ (2c). Suppose D is algebraic and $\mathbf{B}[D]$ has property M. Then $\mathcal{B}$ is a basis for ΣD by (1). If u and v are finite subsets of $\mathbf{B}[D]$ then $\uparrow u \cup \uparrow v = \uparrow w$ where

$$w = \bigcup\{\mathrm{MUB}\{x, y\} \mid x \in u,\ y \in v\}.$$

But w is finite, so $\uparrow w \in \mathcal{B}$. Hence $\mathcal{B}$ is closed under finite intersections. Thus (2c) $\Rightarrow$ (2a).

(3) First we show that (3a) $\Rightarrow$ (3b). By part (1), D is algebraic. Suppose $u \subseteq \mathbf{B}[D]$ is finite. Then $\Im = \{\uparrow x \mid x \in u\}$ is a finite subset of $\mathcal{B}_1$. Suppose $\Im' \subseteq \mathcal{B}_1$ is finite, quasi-conjunctive and $\Im \subseteq \Im'$. There is a finite set $u' \subseteq \mathbf{B}[D]$ such that $u \subseteq u'$ and $\Im' = \{\uparrow x \mid x \in u'\}$. Since $\Im'$ is quasi-conjunctive, we have $u' \triangleleft \mathbf{B}[D]$ because of Lemma 60. This shows that $\mathbf{B}[D]$ is a Plotkin order, so D is profinite. To see that (3b) $\Rightarrow$ (3a), suppose D is profinite and $\Im \subseteq \mathcal{B}_1$ is finite. Then $\Im = \{\uparrow x \mid x \in u\}$ for a finte $u \subseteq \mathbf{B}[D]$. Since D is profinite, $\mathbf{B}[D]$ is a Plotkin order so $u \subseteq u' \triangleleft \mathbf{B}[D]$ for a finite u'. Thus $\Im' = \{\uparrow x \mid x \in u'\}$ is a finite quasi-conjunctive subset of $\mathcal{B}_1$ with $\Im \subseteq \Im'$.

(4) Left to the reader. □

Lemma 62 *If D is algebraic and $O \subseteq D$ is open then O is an algebraic cpo and $\mathbf{B}[O] = \mathbf{B}[D] \cap O$.*

Proof. That O is a cpo is immediate from the fact that it is upward closed and D s a cpo. Let $x \in O$ and set $M = \mathbf{B}[D] \cap \downarrow x$. Since D is algebraic, $\bigsqcup M = x$. But O is open so $O \cap M \neq \emptyset$. If $y \in O \cap M$ then $N = M \cap \uparrow y$ is directed and $\bigsqcup N = x$. Since the elements of N are finite in D they are also finite in O. Hence $\mathbf{B}[D] \cap O$ forms a basis for O. If $x \in \mathbf{B}[O]$ then $x = \bigsqcup M$ for some directed $M \subseteq \mathbf{B}[D] \cap O$. But x is finite so $x \in M$. Hence $\mathbf{B}[O] \subseteq \mathbf{B}[D] \cap O$. The reverse inclusion is obvious so we must have $\mathbf{B}[O] = \mathbf{B}[D] \cap O$. □

Corollary 63 *A compact open subset of a profinite domain is profinite.*

Proof. Suppose D is profinite and $K \subseteq D$ is compact open. Then $K = \uparrow u$ for some finite $u \subseteq \mathbf{B}[D]$ by Lemma 58. Also, by Lemma 62, K is an algebraic cpo with $\mathbf{B}[K] = \mathbf{B}[D] \cap K$. Now, suppose $v \subseteq \mathbf{B}[K]$ is finite. Since $\mathbf{B}[D]$ is a Plotkin order, there is a finite set A such that $u \cup v \subseteq A \triangleleft \mathbf{B}[D]$. We claim that $B = A \cap K \triangleleft \mathbf{B}[K]$. Suppose $x \in \mathbf{B}[K]$. Since $A \triangleleft \mathbf{B}[D]$, there is a largest element x' in $A \cap \downarrow x$. Since $x \in \uparrow u$, $u \subseteq A$ and K is upward closed, we must have $x' \in \mathbf{B}[K]$. Thus x' is in B and we conclude that $B \subseteq \mathbf{B}[K]$. This shows that $\mathbf{B}[K]$ is a Plotkin order so K is profinite. □

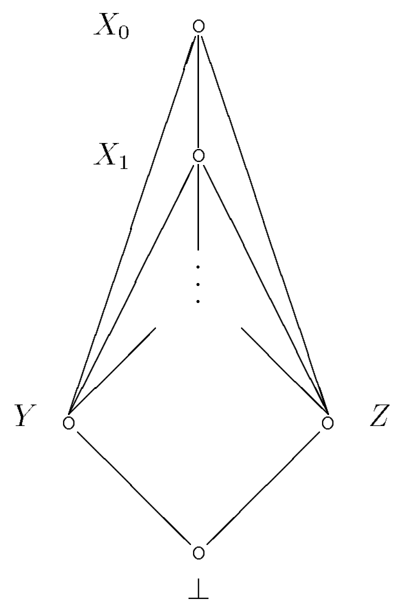


Figure 4.1: A Scott compact poset which is not Lawson compact.

4.5 The Lawson topology

Definition: Let D be a cpo. The *Lawson topology* ΛD on D has a sub-basis Scott open sets and sets of the form $D - \uparrow x$ where x is an arbitrary element of D. □

Let D be an algebraic cpo and suppose $\mathbf{B}[D]$ has property M. We show that the collection $\mathcal{B}$ of sets of the form $\uparrow x - \uparrow u$ where $x \in \mathbf{B}[D]$ and $u \subseteq \mathbf{B}[D]$ is finite form a basis for the Lawson topology on D. Since $\uparrow x$ is Scott open for each $x \in \mathbf{B}[D]$, such sets are certainly Lawson open. To see that any of the sets in the sub-basis of the Lawson topology given above can be written as a union of sets in $\mathcal{B}$, suppose U is Scott open and $x \in U$. Then $x = \bigsqcup \mathbf{B}[D] \cap \downarrow x$ so $U \cap \mathbf{B}[D] \cap \downarrow x \neq \emptyset$. Hence there is a finite $y \in U$ such that $x \in \uparrow y$. Thus any Scott open set is a union of elements of $\mathcal{B}$. Now suppose $U = D - \uparrow x$ and $y \in U$ is arbitrary. Since D is algebraic, there must be a $z \in \mathbf{B}[D]$ which lies below x but not below y. Let z' be any finite element below y and consider the set $V = \uparrow z' - \uparrow \mathrm{MUB}\{z, z'\}$. Since $\mathbf{B}[D]$ has property M, the set $\mathrm{MUB}\{z, z'\}$ is a finite subset of $\mathbf{B}[D]$ and $x \in \uparrow \mathrm{MUB}\{z, z'\}$. Hence $y \in V \subseteq U$. This shows that $\mathcal{B}$ is a sub-basis for the Lawson topology. To see that these sets actually form a basis, suppose $U = \uparrow x - \uparrow u$ and $V = \uparrow y - \uparrow v$ are in $\mathcal{B}$. We show that $U \cap V$ can be written as a union of sets in $\mathcal{B}$. Let

$$w = \bigcup \{\mathrm{MUB}\{x, y, z\} \mid z \in u \cup v\}$$

and set $W = \{\uparrow z - \uparrow w \mid z \in \mathrm{MUB}\{x, y\}\}$. If $a \in U \cap V$ then $a \in \uparrow z$ for some $z \in \mathrm{MUB}\{x, y\}$ by property M and $a \notin \uparrow z$ for any $z \in w$. Thus $U \cap V \subseteq W$. If $a \in W$ and $a \in \uparrow u$ then $a \sqsupseteq z$ for some $z \in u$ so $a \sqsupseteq z'$ for some $z' \in \mathrm{MUB}\{x, y, z\}$. But then $a \notin W$. So $a \in W$ implies $a \in \uparrow x - \uparrow u$. Hence $W \subseteq U$. A similar argument show that $W \subseteq V$. We conclude that $W = U \cap V$ and $\mathcal{B}$ is therefore a basis for D.

Lemma 64 *Suppose $\Delta = \langle A_i, a_{ij}\rangle_{i\in I}$ is an inverse system of finite posets in* $\mathbf{CPO}^P$. *Let each A_i be given the discrete topology and give $\prod_i A_i$ the product topology. Then the induced topology on $\varprojlim \Delta$ considered as a subset of $\prod_i A_i$ coincides with the Lawson topology on $\varprojlim \Delta$ considered as a cpo.*

Proof. It simplifies matters to assume that $A_i \lhd A_j$ for each $i \leq j$ and define $a_{ij}(x) = \bigsqcup\{y \in A_j \mid x \sqsupseteq y\}$. We show first that if $n \in I$ and

$$U = \prod_{i\leq n} O_i \times \prod_{i>n} A_i$$

then $U \cap \varprojlim \Delta$ is open in the Lawson topology on $\varprojlim \Delta$. Since sets having the form of U provide a basis for the product topology on $\prod_i A_i$ this will show that the induced topology is finer than the Lawson topology. Let

$$S = \{x \in O_n \mid a_{nm}(x) \in O_m \text{ for each } m \leq n\}$$

and for each $x \in S$, let

$$U_x = \uparrow\bar{x} - \bigcup\{\uparrow\bar{y} \mid y \in A_n \text{ and } y \sqsupseteq x \text{ but } y \notin S\}$$

where $\bar{x}$ abbreviates $a_{i*}(x)$. Since A_n is finite, each U_x is Lawson open. Note that for $x \in \varprojlim \Delta$, $x \in U$ if and only if $x_n \in S$. Now, if $x \in S$ and $y \in U_x$ then $y_n \in S$ so $y \in U$. Hence $U_x \subseteq U$ for each $x \in S$. Suppose $x \in U \cap \varprojlim \Delta$. Then $x_n \in S$ so $x_n \in U_{x_n}$. Hence $U \cap \varprojlim \Delta$ is equal to the union of the sets U_x such that $x \in S$ and is therefore open in the Lawson topology. To prove that a Lawson open subset of $\varprojlim \Delta$ is open in the induced topology, suppose $x, y^1, \ldots, y^k \in A_n$ for some n and consider the set

$$U = \uparrow\bar{x} - (\uparrow\bar{y}^1 \cup \cdots \cup \bar{y}^k) \subseteq \varprojlim \Delta.$$

Let $O = \{z \in A_n \mid z \sqsupseteq x \text{ but } z \not\sqsupseteq y^i \text{ for } i \leq k\}$ and set $V = \prod_i O_i$ where $O_i = A_i$ for $i \neq n$ and $O_n = O$. Then $z \in V$ if and only if $z_k \in O$ if and only if $x \in U$. Since V is open in the product topology and sets like U form a basis for the Lawson topology on $\varprojlim \Delta$, we are done. □

Corollary 65 *The Lawson topology on a profinite domain is a Stone space, i.e. if D is profinite then ΛD is compact, Hausdorff and zero dimensional.*

Proof. Let Δ be a $\mathbf{CPO}^P$ inverse system of finite posets and suppose $x \in \prod_i A_i - \varprojlim \Delta$. Then, for some k and j, $a_{kj}(x_k) \neq x_j$. So let $O_k = \{x_k\}$ and $O_j = \{x_j\}$. Then $x \in U = \prod_i O_i$ where $O_i = A_i$ for each $i \neq k, j$. Moreover, $U \cap \varprojlim \Delta = \emptyset$. Since U is open, this shows that $\prod_i A_i$ is a closed subspace of $\varprojlim \Delta$. But $\prod_i A_i$ is a Stone space and a closed subspace of a Stone space is itself a Stone space. Hence, by Lemma 58 the Lawson topology on a profinite domain is a Stone space. □

Of course, if a set is not Scott compact, then it is not Lawson compact. However, it is not true that any Scott compact cpo is also Lawson compact. A counter-example appears in Figure 4.1. Let D be the poset pictured there. Since D has the acc, all of its members are finite, so for each $i \in \omega$ the set $\uparrow X_i$ is Scott open. Moreover, the sets $D - \uparrow Y$ an $D - \uparrow Z$ are Lawson open. Although these sets cover D, no finite subset of them will do so. It is *not* true that an algebraic cpo which is Lawson compact is profinite. For example the ideal completion of the poset pictured in Figure 2.1c is a Lawson compact algebraic cpo but it is not profinite.

Chapter 5

Universal Domains

All of the existing approaches to the solution of recursive domain equations use one of three techniques. Perhaps the most general is the inverse limit construction used by Scott [1972] to solve the domain equation $D \cong [D \to D]$ (where $[D \to D]$ is the function space of D). A second technique uses the Tarski Fixed Point Theorem, which says: if D is a cpo with a least element then any continuous function $f : D \to D$ has a least fixed point. The third—which is introduced in [MacQueen *et. al.* 1984]—uses the Banach Fixed Point Theorem, which says: a uniformly contractive function $f : X \to X$ on a non-empty complete metric space X has a unique fixed point. These last two approaches employ what are generally called "universal domains" to associate with the operator F a lub preserving or contractive map. In this chapter we introduce a variant on the second technique which can be applied to certain endofunctors on the ω-profinite domains.

5.1 Universal models in logic and domain theory

We now investigate the mathematical problem of the existence of a universal domain in the category of countably based profinite posets with embeddings as arrows. The term "universal domain" will be used here in the sense that it is used by researchers in domain theory. One might describe the idea categorically as follows. A universal object for a category **C** is a **C**-object A such that for every **C**-object B there is a (not necessarily unique) arrow form B to A. Of course, the interest of a **C**-universal object depends on the objects and morphisms of **C**. For example, in the category of groups and homomorphisms *every* object is universal. But if **C** is the category of countable groups and injective homomorphisms then the existence or non-existence of a universal object for **C** is less obvious. On cannot, say, produce a universal group by taking a product of cannonical representatives of all the isomorphism classes of countable groups since there are continuum many such classes. But the existence of continuum many isomorphism classes does not in itself rule out the possibility of there being a countable universal group. In fact, there is no such group and we leave it to the reader (especially if he or she is a group theorist) to find a reason.

Perhaps the best examples of universal objects and best general techniques for finding them (or proving that they do not exist) can be found in the literature on first order model theory. One of the most well known examples of a universal object in model theory is the

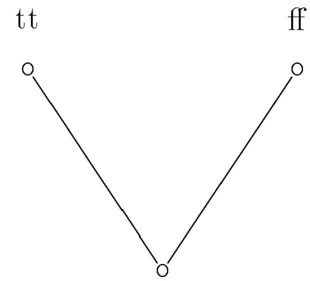


Figure 5.1: The truth value cpo.

order type of rational numbers Q which is universal for the category of countable linear orders and injective monomorphisms. The most elegant proof of this universality uses a technique known to logicians as a "forth" construction. [1] In short the proof proceeds as follows. Let L be a countable linear order and suppose $X_1, X_2, \ldots$ is an enumeration of L such that the X_i's are distinct. Let Y_1 be any element of Q and suppose we have defined $Y_1, \ldots, Y_n \in Q$ so that for each $i, j \leq n$, $X_i <_L X_j$ if and only if $Y_i < Y_j$. Then because Q is dense we can find a $Y_{n+1} \in Q$ such that $Y_j < Y_{n+1} < Y_i$ for $i, j \leq n$ if and only if $X_j <_L X_{n+1} <_L X_i$. We iterate this operation *ad infinitum* and the correspondence $X_n \mapsto Y_n$ then defines the desired injective monomorphism.

One might say that this construction works because Q is dense. In the language of model theory, it works because Q is *saturated*. In fact, whenever T is a first order theory in a countable language, if A is an ω-saturated countable model of T then A is universal for countable models of T and elementary embeddings. Such a model is called "countably universal" in [Chang and Keisler 1973] and is a special instance of the idea of universality we mentioned above. In what follows the notion of universality which interests us is less restrictive in that it will not require so much of the embeddings. However, the class of models considered is too complex to be the class of countable models of a first order theory. Hence the first order methods of constructing suitably saturated models via, for example, elementary chains cannot be employed directly. But the analogy with the methods that we do use will be evident.

In the literature there are three primary examples of universal domains. The simplest is the so-called *graph model* $\mathcal{P}\omega$ which is the algebraic lattice of subsets of ω ordered by set inclusion. It recieves a detailed study in [Scott 1976] where it is proved that any countably based algebraic lattice is a continuous retract of $\mathcal{P}\omega$. Some domain theorists felt, however, that for applications in denotational semantics it would be easier to use a class which did not require the existence of a largest (top) element. Plotkin [1978b] showed that the poset T^ω of functions from ω into the truth value cpo T (see figure 5.1 is universal in the sense that every coherent ω-algebraic cpo is a continuous retract of T^ω. Since T^ω is itself algebraic and coherent this provided a universal domain for a class of algebraic cpo's that included the algebraic lattices but contained also certain desired cpo's without tops.

In [Scott 1981a, 1981b, and 1982a] yet a third universal domain $\mathcal{U}$ is discussed. Although $\mathcal{U}$ is harder to understand than $\mathcal{P}\omega$ or T^ω it has the advantage of having every consistently

[1]Because it is "half"of the "back and forth" construction which is so frequently used to demonstrate the uniqueness of a model up to isomorphism.

complete ω-algebraic cpo as a continuous *projection* (not just as a retract). Elementary proofs of this fact appear in [Scott 1981a] and in [Bracho 1983]. A less elementary proof can be carried out as follows. Let B be the countable atomless boolean algebra and suppose A is a countable consistently complete poset. Now, A can be embedded into a countable boolean algebra in a way that preserves existing joins in A and such that the join of the image of an unbounded subset of A is the top element. But any countable boolean algebra is isomorphic to a subalgebra of B. Thus $A \mathrel{\underset{\sim}{\lhd}} B^-$ where B^- is B minus its top element. We conclude that if A is countable and consistently complete then there is a continuous projection $p : |B^-| \to |A|$. Thus $\mathcal{U} = |B^-|$ is universal for the consistently complete algebraic cpo's.

In what follows we use a technique similar to the one for $\mathcal{U}$ to get universal domains for certain classes of ω-profinite domains. To explain the result, recall that if A is a Plotkin order, poset then the root of A is the smallest normal substructure of A. Now, if A and B are Plotkin posets and $A \mathrel{\underset{\sim}{\lhd}} B$ then $\mathrm{rt}(A) \cong \mathrm{rt}(B)$ so no profinite domain can be a continuous projection of a profinite domain that has a different root. Hence there cannot be a projection universal ω-profinite domain. We prove the next best thing: for each poset $A \cong \mathrm{rt}(A)$ there is a countable Plotkin poset V_A such that if B is a countable Plotkin poset with $\mathrm{rt}(B) \cong A$ then $B \mathrel{\underset{\sim}{\lhd}} V_A$.

These models are less elegant than $\mathcal{P}\omega$, T^ω or $\mathcal{U}$ because they are *built* to be universal. In other words, what we have is not so much a model as a *technique* for generating a model. Full details of one technique of construction are offered here and we mention another (closely related) technique at the end. Kamimura and Tang [1984] use a different approach to get a retraction universal model for the ω-profinite domains having bottoms. Their model, like $\mathcal{P}\omega$ and T^ω, is locally finite but is somewhat less natural than either of those models. In the opinion of the author, however, the construction described below does the most to reveal the fundamental *idea* that gives the existence result and yields the most detailed description of the model being built. (We are even able to draw a partial picture of it!)

5.2 Universal profinite domains

We begin by proving an interesting structure theorem for Plotkin posets.

Proposition 66 *If A and B are finite posets such that $A \lhd B$ but $A \neq B$ then there are posets $A_0, \ldots, A_n$ such that*

$$A = A_0 \lhd A_1 \lhd \cdots \lhd A_{n-1} \lhd A_n = B$$

and for each $k < n$, $A_{k+1} - A_k$ is a singleton.

Proof. If $B - A$ is a singleton then we are done. Assume that the result holds for any pair $A' \lhd B'$ such that $B' - A'$ has fewer that n elements. Suppose there are n elements in $B - A$ and let X be a maximal element of $B - A$, *i.e.* if $Y \in B$ such that $X \sqsubset Y$ then $Y \in A$. Set $A' = A \cup \{X\}$. We show that $A' \lhd B$. Let $Z \in B$ and suppose

$$u = \{Y \in A' \mid Y \sqsubseteq Z\}.$$

We must demonstrate that u has a largest element. If $u \subseteq A$ then this follows from the fact that $A \lhd B$. if $X \in u$ then $X \sqsubseteq Z$ so $X = Z$ or $Z \in A$. In either case, Z is the largest element of u. Hence $A' \lhd B$. Since $A \lhd B$ we have $A \lhd A' \lhd B$. But $B - A'$ has $n-1$ elements, so by the induction hypothesis, there are posets $A_1, \ldots, A_n$ such that

$$A \lhd A' = A_1 \lhd \cdots \lhd A_n = B.$$

□

Theorem 67 *(Enumeration) If A is a countable Plotkin poset and $B = \mathrm{rt}(A)$, then there is an enumeration $X_0, X_1, \ldots$ of A such that for each n,*

$$B \cup \{X_i \mid i < n\} \lhd A.$$

Proof. Suppose $\mathrm{rt}(A) = A_0 \lhd A_1 \lhd \cdots$ is a chain of finite normal substructures of A such that $A = \bigcup_{n \in \omega} A_n$. Let $B_0 \lhd B_1 \lhd \cdots$ be a new chain that results from deleting A_{n+1} for each n if it equals A_n. Using Lemma 66 we may refine this chain to a chain $C_0 \lhd C_1 \lhd \cdots$ such that $C_0 = \mathrm{rt}(A)$ and for each n, $C_{n+1} - C_n$ is a singleton Z_n. Now, let $X_0, \ldots, X_{k-1}$ be an enumeration of C_0 and for each n, let $X_{n+k} = Z_n$. This enumeration has the desired properties. □

Definition: Let $\langle A, \sqsubseteq \rangle$ be a poset. For each $X \in A$, let $\mathbf{X}$ be a constant symbol naming X. Let $\preceq$ be a binary relation symbol which is interpreted by $\sqsubseteq$. A *diagram type* over A is a set Γ of inequalities and negations of inequalities between constant symbols and a variable $\mathbf{v}$, *i.e.* formulas of the form

$$\mathbf{v} \preceq \mathbf{X}, \qquad \mathbf{v} \not\preceq \mathbf{X}, \qquad \mathbf{X} \preceq \mathbf{v}, \qquad \mathbf{X} \not\preceq \mathbf{v}$$

where $X \in A$. If $A \subseteq B$ and $Z \in B$ then the *diagram type of Z over A* is the set of all such equations (using constant symbols for elements of A) that hold when $\mathbf{v}$ is given the value Z and $\preceq$ is interpreted as the order relation on B. A diagram type Γ over A is said to be *realized* in B by Z if Γ is a subset of the diagram type of Z over A. A diagram type Γ over a poset A is said to be *normal* if there is a poset B with $A \lhd B$ such that Γ is realized in B. □

Lemma 68 *If Γ is a normal type over a finite poset B and $B \lhd A$ then there is a finite poset A_1 such that $A \lhd A_1$ and Γ is realized by some $Z \in A_1$ such that $B \cup \{Z\} \lhd A_1$.*

Proof. Let $\sqsubseteq$ be the partial ordering on A. Since $B \lhd A$, B inherits this ordering. Suppose $B \lhd A_0$ and $Z \in A_0$ such that Z realizes Γ. Let $\sqsubseteq_0$ be the partial ordering on A_0. Note that the restriction of $\sqsubseteq_0$ to B is the same as the restriction of $\sqsubseteq$ to B. Let $A_1 = A \cup \{Z\}$ and define a binary relation $\sqsubseteq_1$ on A_1 as follows:

- $Z \sqsubseteq_1 Z$,
- if $X, Y \in A$ then $X \sqsubseteq_1 Y$ iff $X \sqsubseteq Y$,
- if $X \in A$ then $X \sqsubseteq_1 Z$ iff there is an $X' \in B$ such that $X \sqsubseteq X' \sqsubseteq_0 Z$,
- if $X \in A$ then $Z \sqsubseteq_1 X$ iff there is an $X' \in B$ such that $Z \sqsubseteq_0 X' \sqsubseteq X$.

To see that $\langle A_1, \sqsubseteq_1 \rangle$ is a poset, note that $\sqsubseteq_1$ is just the transitive closure of $(\sqsubseteq \cup \sqsubseteq_0) \cap (A_1 \times A_1)$. That $\sqsubseteq_1$ is reflexive is immediate from its definition. To see that it is anti-symmetric, suppose $X \sqsubseteq_1 Z \sqsubseteq_1 X$ for some $X \in A$. Then there are $X_0, X_1 \in B$ such that $X \sqsubseteq X_0 \sqsubseteq_0 Z$ and $Z \sqsubseteq_0 X_1 \sqsubseteq X$. But then $X \sqsubseteq X_0 \sqsubseteq X_1 \sqsubseteq X$ so $X_0 = X_1 = X$ and therefore $X \in B$. Hence $X \sqsubseteq_0 Z \sqsubseteq_0 X$ implies $X = Z$ by the anti-symmetry of $\sqsubseteq_0$. Of course, if $X, Y \in A$ and $X \sqsubseteq_1 Y \sqsubseteq_1 X$ then $X = Y$ since $X \sqsubseteq Y \sqsubseteq X$.

Now, the fact that A is a sub-poset of A_1 is built into the definition of $\sqsubseteq_1$ by 2. To see that $A \triangleleft A_1$, suppose $u \subseteq A$ is finite and $u \sqsubseteq_1 Z$. By the definition of $\sqsubseteq_1$, for each $X \in u$ there is an $X' \in B$ such that $X \sqsubseteq X' \sqsubseteq_1 Z$. So let $u' = \{X' \mid X \in u\}$. Then $u' \sqsubseteq Z$. Since $B \triangleleft A_0$, there is a $Z' \in B$ such that $u' \sqsubseteq_0 Z' \sqsubseteq_0 Z$. But this implies that $u \sqsubseteq_1 Z' \sqsubseteq_1 Z$ so we may infer that $A \triangleleft A_1$. We must show that $B \cup \{Z\} \triangleleft A_1$. Suppose $u \subseteq B \cup \{Z\}$ is finite and $u \sqsubseteq_1 X$ for some $X \in A_1$. We must find a $Y \in B \cup \{Z\}$ such that $u \sqsubseteq_1 Y \sqsubseteq_1 X$. If $X = Z$ then the result is immediate. So suppose $X \in A$. If $Z \notin u$ then we can get the desired Y by using the fact that $B \triangleleft A$. If $Z \in u$ then there is an $X' \in B$ such that $Z \sqsubseteq_0 X' \sqsubseteq X$. Thus

$$v = (u - \{Z\}) \cup \{X'\} \sqsubseteq X.$$

Since $B \triangleleft A$ and $v \subseteq B$, there is some $Y \in B$ such that $v \sqsubseteq Y \sqsubseteq X$. Since $Z \sqsubseteq_0 X' \sqsubseteq Y$ we may conclude that $Z \sqsubseteq_1 Y$. Thus $u \sqsubseteq_1 Y$ and we are done.

Finally, suppose $\mathbf{v} \preceq \mathbf{X}$ is in Γ for some $X \in B$. Then $Z \sqsubseteq_0 X$ since Z realizes Γ in A_0. Hence, by definition, $Z \sqsubseteq_1 X$. Suppose $\mathbf{v} \not\preceq \mathbf{X}$ is in Γ but $Z \sqsubseteq_1 X$. Then $Z \sqsubseteq_0 X$. But this contradicts the assumption that Z realizes Γ in A_0. So apparently $Z \not\sqsubseteq_1 X$. Similarly, the other formulas in Γ must be realized by Z in A_1. □

Lemma 69 *Let A be a finite poset. Then there is a finite poset A^+ such that $A \triangleleft A^+$ and for every subspace $B \triangleleft A$ and normal type Γ over B, there is a $Z \in A^+$ such that Z realizes Γ and $B \cup \{Z\} \triangleleft A^+$.*

Proof. Let $\Gamma_1, \ldots, \Gamma_n$ be all of the normal types over normal subspaces of A. Set $A = A_0$ and suppose $A \triangleleft A_k$. Suppose Γ_{k+1} is normal over $B \triangleleft A$. Then $B \triangleleft A_k$ so by Lemma 68 there is a finite poset A_{k+1} such that $A_k \triangleleft A_{k+1}$ and $B \cup \{Z\} \triangleleft A_{k+1}$ for some Z that realizes Γ_{k+1}. Set $A^+ = A_{n+1}$. If Z realizes Γ_{k+1} in A_{k+1} then it realizes it also in A^+. Moreover, $B \cup \{Z\} \triangleleft A_{k+1} \triangleleft A^+$. □

Theorem 70 *Let V be a countable Plotkin poset. Suppose that for every finite $A \triangleleft V$ and normal type Γ over A, there is a realization Z for Γ such that $A \cup \{Z\} \triangleleft V$. If B is a countable Plotkin order such that $\mathrm{rt}(B) \cong \mathrm{rt}(V)$ then $B \trianglelefteq V$.*

Proof. Suppose B is a countable Plotkin order such that $\mathrm{rt}(B) \cong \mathrm{rt}(V)$. We may assume that B is a poset. By Proposition 67, there is an enumeration $X_0, X_1, \ldots$ of B such that for each $n \in \omega$,

$$B_n = \mathrm{rt}(B) \cup \{X_i \mid i < n\} \triangleleft B.$$

Since $B_0 = \mathrm{rt}(B)$, there is an isomorphism $f_0 : B_0 \cong V_0$ where $V_0 = \mathrm{rt}(V)$. We construct a sequence of isomorphisms f_n such that for each $n \in \omega$, $f_n : A_n \cong V_n$ where $V_n \triangleleft V$, $f_n \subseteq f_{n+1}$ and $V_n \subseteq V_{n+1}$.

Suppose that f_n and V_n are given. Now, $B_n \lhd B_{n+1}$ so the diagram type Γ of X_n over B_n must be normal. Let Σ be the corresponding type over V_n, *i.e.* Σ is obtained from Γ by replacing any occurrence of a constant symbol for an $X \in A_n$ by a constant symbol for $f_n(X)$. Then Σ is a normal type over V_n so by the hypothesis on V, there is a realization $Y_n \in V$ of Σ such that

$$V_{n+1} = V_n \cup \{Y_n\} \lhd V.$$

If we define $f_{n+1} : A_{n+1} \to V_{n+1}$ by

$$f_{n+1}(X) = \begin{cases} f_n(x) & \text{if } X \in A_n; \\ Y_n & \text{if } X = X_n, \end{cases}$$

then $f_n \subseteq f_{n+1}$ and f_{n+1} is an isomorphism. If $f = \bigcup_{n\in\omega} f_n$ and $V' = \bigcup_{n\in\omega} V_n$ then $f : B \cong V'$. Moreover, since $V_n \lhd V$ for each n, $V' \lhd V$. Hence $B \precsim V$. □

Corollary 71 *Let A be a finite poset such that $A \cong \mathrm{rt}(A)$. There is a Plotkin poset $A^\star$ such that whenever B is a countable Plotkin poset with $\mathrm{rt}(B) \cong A$, then $B \precsim A^\star$.*

Proof. Let $A = A_0$ and for each n, define $A_{n+1} = A_n^+$. Let $A^\star = \bigcup_{n\in\omega} A_n$. Suppose $C \lhd A^\star$ is finite. Then $C \lhd A_n$ for some n. If Γ is a normal type over C then Γ is realized by a $Z \in A_n^+ = A_{n+1}$ such that $C \cup \{Z\} \lhd A_{n+1}$. Since $A_{n+1} \lhd A^\star$, the hypotheses of Theorem 70 are satisfied and the desired conclusion therefore follows. □

It is possible to get the A^+ in Lemma 69 by explicit construction. One way to do this is to pre-order the set

$$T = \{\Gamma \mid \ \Gamma \text{ is normal over some finite } B \lhd A\}$$

by letting $\Gamma \vdash \Sigma$ just in case there are $X, Y \in A$ such that $\mathbf{v} \preceq \mathbf{X}$ is in Γ, $\mathbf{Y} \preceq \mathbf{v}$ is in Σ, and $X \sqsubseteq Y$. If we let $A^+ = \tilde{T}$ then there is a normal substructure $A' \lhd A^+$ with $A \cong A'$ such that for every normal type Γ over a substructure $B \lhd A'$, there is a $Z \in A^+$ such that $B \cup \{Z\} \lhd A^+$ and Z realizes Γ. To get a universal domain one merely solves the domain equation $A = A^+$. Although it is somewhat tedious to check all of the details of the construction, this more order-theoretic way of doing things helps in picturing the universal domain as the limit of the posets $A \precsim A^+ \precsim A^{++} \precsim \cdots$. Figure 5.2 illustrates the first three stages in the construction of the universal domain with a trivial root.

5.3 Using universal domains to obtain fixed points

Let $\omega\mathbf{RP}$ be the category of continuous retracts of ω-profinite domains with continuous functions as arrows. In this section we show that a significant class of endofunctors on $\omega\mathbf{RP}$ possess fixed points. The argument we give uses the universal domains defined above and specializes to a proof that if such a functor is also an endofunctor on $\omega\mathbf{P}$ then it has an ω-profinite fixed point.

Theorem 72 *Let D be a cpo. The following are equivalent:*

Figure 5.2: Construction of $1^\star$.

1. *D is a continuous retract of a countably based profinite domain.*

2. *D is a continuous projection of a countably based profinite domain.*

3. *There is an ω-sequence of continuous functions $f_i : D \to D$ such that for each $i, j \in \omega$,*

   (a) $i \leq j$ implies $f_i \sqsubseteq f_j$,

   (b) $im(f_i)$ is finite,

   (c) $\bigsqcup_{i\in\omega} f_i = \mathrm{id}_D$.

Proof. Since a projection is a retraction we certainly have (2) $\Rightarrow$ (1). We show that (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

(1) $\Rightarrow$ (3). Suppose E is ω-profinite and there are continuous functions $r : E \to D$ and $r' : D \to E$ such that $r \circ r' = \mathrm{id}_D$. Since E is ω-profinite, there is a sequence $\langle p_i \rangle_{i\in\omega}$ of finite deflations on E such that $p_i \sqsubseteq p_j$ whenever $i \leq j$ and $\bigsqcup p_i = \mathrm{id}_E$. For each $i \in \omega$, define a continuous function $f_i = r \circ p_i \circ r' : D \to D$. If $i \leq j$ then $f_i = r \circ p_i \circ r' = r \circ p_j \circ r' = f_j$. Moreover,

$$\bigsqcup_{i\in\omega} f_i = \bigsqcup_{i\in\omega} r \circ p_i \circ r' = r \circ (\bigsqcup_{i\in\omega} p_i) \circ r' = r \circ r' = \mathrm{id}_D.$$

Finally, $im(f_i)$ is finite for each i because $im(p_i)$ is. Thus the sequence $\langle f_i \rangle_{i\in\omega}$ satisfies (a), (b) and (c).

(3) $\Rightarrow$ (2). Suppose D is a cpo and $\langle f_i \rangle_{i\in\omega}$ is a sequence of functions satisfying conditions (a), (b) and (c). Let E be the set of *monotone* sequences $x : \omega \to D$ such that for each $i \in \omega$,

$x_i \in F_i = \bigcup_{j \leq i} im(f_j)$ and

$$x_i \sqsupseteq f_i(\bigsqcup_{j \in \omega} x_j). \qquad (*)$$

Order E coordinatewise, *i.e.* $x \sqsupseteq y$ if and only if $x_i \sqsupseteq y_i$ for each i. We claim that E is profinite. To see that E is a cpo suppose $M \subseteq E$ is directed. We show that the least upper bound x of M in $\prod_{i \in \omega} F_i$ is in E. Now, x is certainly monotone; to prove that x satisfies condition $(*)$, we calculate

$$\begin{aligned} f_i(\bigsqcup_{j \in \omega} x_j) &= f_i(\bigsqcup_{j \in \omega} \bigsqcup \{y_j \mid y \in M\}) \\ &= f(\bigsqcup \{\bigsqcup_{j \in \omega} y_j \mid y \in M\}) \\ &= \bigsqcup \{f_i(\bigsqcup_{j \in \omega} y_j) \mid y \in M\} \\ &= \bigsqcup \{y_i \mid y \in M\} && \text{by } (*) \text{ for } y \in M. \\ &= x_i \end{aligned}$$

Let $A \subseteq E$ be the set of sequences $x \in E$ such that for some n,

$$\forall i \geq n.\ x_i = x_n. \qquad (**)$$

We claim that A is a basis of finite elements for E. Suppose x and n have property $(**)$ and $M \subseteq E$ is directed with $\bigsqcup M = x$. Since F_n is finite there is some $y \in M$ such that $y_n = x_n$. Since $y_i \geq y_n$ for each $i \geq n$ we must have $x_i = y_i$ for each $i \geq n$. The set of $z \in M$ such that $z \sqsupseteq y$ is therefore finite. Hence $x \in M$ so $x \in \mathbf{B}[E]$. Now, let

$$A_n = \{x \in E \mid \forall i \geq n.\ x_i = x_n\}.$$

Suppose $u \subseteq A_n$ and $x \in E$ such that $x \sqsupseteq u$. Define a sequence x by

$$x'_i = \begin{cases} x_i & \text{if } i \leq n; \\ x_n & \text{otherwise.} \end{cases}$$

Now, for each $i \leq n$, $x'_i = x_i \sqsupseteq f_i(\bigsqcup_{j \in \omega} x_j) \sqsupseteq f_i(\bigsqcup_{j \in \omega} x'_j)$. If $i \geq n$ then $x'_i = x_n \sqsupseteq f_i(x_n) = f_i(\bigsqcup_{j \in \omega} x_j)$. Hence x' is in E. If $y \in u$ then $x'_i = x_i \sqsupseteq y_i$ for each $i \leq n$ and if $i \geq n$ then $x'_i = x_n \sqsupseteq y_n = y_i$. Thus $x \sqsupseteq x' \sqsupseteq u$ and we conclude that $A_n \triangleleft E$. Now, A_n is finite for each n and $A_n \triangleleft A_m$ whenever $n \leq m$. Since $A = \bigcup_{n \in \omega} A_n$ we conclude that A is a Plotkin order. Moreover, it is obvious that for any $x \in E$, $x = \bigsqcup \{y \in A \mid x \sqsupseteq y\}$. Hence $A = \mathbf{B}[E]$ is countable and E is profinite. To complete the proof, define $p : E \to D$ by $p : x \mapsto \bigsqcup_{j \in \omega} x_j$ and $q : D \to E$ by $q : x \mapsto \langle f_i(x) \rangle_{i \in \omega}$. It is easy to check that p and q are continuous. If $x \in D$ then $(p \circ q)(x) = \bigsqcup_{i \in \omega} f_i(x) = x$. If $x \in E$ then $(q \circ p)(x) = q(\bigsqcup_{j \in \omega} x_j) = \langle f_i(\bigsqcup_{j \in \omega} x_j) \rangle_{i \in \omega} \sqsubseteq \langle x_i \rangle_{i \in \omega}$. Hence D is the continuous projection of a countably based profinite domain. □

Corollary 73 *For a cpo D, let* Defl(D) *be the poset of deflations on D. Then* Defl(D) *is a cpo, and if D is in ω***RP** *then* Defl(D) *has a least element.*

Proof. Suppose $M \subseteq \mathrm{Defl}(D)$ is directed and let $p = \bigsqcup M$. Then

$$\begin{aligned} p \circ p &= (\textstyle\bigsqcup M) \circ (\bigsqcup M) \\ &= \textstyle\bigsqcup \{ f \circ g \mid f, g \in M \} \\ &= \textstyle\bigsqcup \{ f \circ f \mid f \in M \} && \text{since } M \text{ is directed} \\ &= \textstyle\bigsqcup M && \text{since } M \subseteq \mathrm{Defl}(D) \\ &= p \end{aligned}$$

and $p \sqsubseteq \mathrm{id}_D$ since $f \sqsubseteq \mathrm{id}_D$ for each $f \in M$. Hence $p \in \mathrm{Defl}(D)$. If D is in $\omega\mathbf{RP}$ then by Theorem 72, D is isomorphic to a normal substructure D' of a profinite domain E. This corresponds—via the isomorphism between D and D'—to a least deflation on D. □

Corollary 74 $\omega\mathbf{RP} \cap \mathbf{ALG} = \omega\mathbf{P}$.

Proof. Suppose D is in $\omega\mathbf{RP} \cap \mathbf{ALG}$. Then it satisfies condition (3) of Theorem 72. Since D is algebraic it satisfies condition (5) of Theorem 37 and is therefore profinite. Since D is a continuous retract of an ω-algebraic cpo it has a countable basis so it is ω-profinite. Hence $\omega\mathbf{RP} \cap \mathbf{ALG} \subseteq \omega\mathbf{P}$. If D is ω-profinite then it is algebraic and a continuous retract of itself. The corollary therefore follows. □

Corollary 75 *A cpo D is ω-profinite if and only if it is in $\omega\mathbf{RP}$ and ΣD has a compact basis.*

Proof. This follows from the corollary above and Theorem 61. □

Definition: A functor $F : \mathbf{CPO} \to \mathbf{CPO}$ is *locally continuous* if it is continuous on hom sets, *i.e.* if $M \subseteq \mathbf{CPO}(D, E)$ is directed for cpo's D and E then $\bigsqcup F(M) = F(\bigsqcup M)$. □

Lemma 76 *Let D be a cpo and $F : \mathbf{CPO} \to \mathbf{CPO}$ a functor. If $r : D \to D$ is a continuous idempotent function then $im(F(r)) \cong F(im(r))$.*

Proof. Let $E = im(r)$ and suppose $i : E \hookrightarrow E$ and $r^\circ : D \to E$ are the inclusion map and corestriction of r respectively. Note that $r = i \circ r^\circ$ and $\mathrm{id}_E = r^\circ \circ i$. Similarly, let $j : E' \hookrightarrow F(D)$ and $F(r)^\circ : F(D) \to E'$ where $E' = im(F(r))$. Then $F(r) = j \circ F(r)^\circ$ and $\mathrm{id}_{E'} = F(r)^\circ \circ j$. Consider the maps

$$\begin{aligned} &F(r)^\circ \circ F(i) : F(E) \to E' \text{ and} \\ &F(r^\circ) \circ j : E' \to F(E). \end{aligned}$$

We claim that these functions are inverse to one another. We have

$$\begin{aligned} (F(r^\circ) \circ j) \circ (F(r)^\circ \circ F(i)) &= F(r^\circ) \circ (j \circ F(r)^\circ) \circ F(i) \\ &= F(r^\circ) \circ F(r) \circ F(i) \\ &= F(r^\circ \circ r \circ i) \\ &= F(r^\circ \circ (i \circ r^\circ) \circ i) \\ &= F((r^\circ \circ i) \circ (r^\circ \circ i)) \\ &= F(\mathrm{id}_E) \\ &= \mathrm{id}_{F(E)} \end{aligned}$$

and

$$\begin{aligned}(F(r)^\circ \circ F(i)) \circ (F(r^\circ) \circ j) &= F(r)^\circ \circ F(i \circ r^\circ) \circ j \\ &= F(r)^\circ \circ F(r) \circ j \\ &= F(r)^\circ \circ (j \circ F(r)^\circ) \circ j \\ &= (F(r)^\circ \circ j) \circ (F(r)^\circ \circ j) \\ &= \mathrm{id}_{E'}.\end{aligned}$$

This demonstrates the desired isomorphism. □

Theorem 77 *Let $F : \omega\mathbf{RP} \to \omega\mathbf{RP}$ be a locally continuous functor and suppose $A \cong \mathrm{rt}(F(A))$ for some finite poset A. Then there is a cpo D in $\omega\mathbf{RP}$ such that $F(D) \cong D$. If F is an endofunctor on $\omega\mathbf{P}$ then F has an ω-profinite fixed point.*

Proof. By Theorems 71 and 72, there is a pair $\langle p, q\rangle : A^\star \xrightarrow{\mathrm{pe}} F(A^\star)$. Define a map

$$d : \mathrm{Defl}(A^\star) \to \mathrm{Defl}(A^\star)$$

by $d : f \mapsto q \circ F(f) \circ p$. To see that this makes sense we must show that $q \circ F(f) \circ p$ is a deflation. First of all, we have

$$\begin{aligned}d(f) \circ d(f) &= (q \circ F(f) \circ p) \circ (q \circ F(f) \circ p) \\ &= q \circ F(f) \circ F(f) \circ p \\ &= q \circ F(f \circ f) \circ p \\ &= d(f)\end{aligned}$$

and by the local continuity of F, $F(f) \sqsubseteq F(\mathrm{id}_{A^\star}) = \mathrm{id}_{F(A^\star)}$ so $d(f) = q \circ F(f) \circ p \sqsubseteq q \circ p \sqsubseteq \mathrm{id}_{F(A^\star)}$. Now, suppose $M \subseteq \mathrm{Defl}(A^\star)$ is directed, then

$$\begin{aligned}d(\bigsqcup M) &= q \circ F(\bigsqcup M) \circ p \\ &= q \circ \bigsqcup F(M) \circ p \\ &= \bigsqcup \{q \circ F(f) \circ p \mid f \in M\} \\ &= \bigsqcup d(M).\end{aligned}$$

Hence d is continuous. Since $\mathrm{Defl}(A^\star)$ is a cpo with a least element and d is continuous, there is an $f \in \mathrm{Defl}(A^\star)$ such that $d(f) = f$. Hence

$$\begin{aligned}im(f) &= im(d(f)) \\ &= im(q \circ F(f) \circ p) \\ &= im(q \circ F(f)) && \text{since } p \text{ is an epimorphism} \\ &\cong im(F(f)) && \text{since } q \text{ is a monomorphism} \\ &\cong F(im(f)). && \text{by Lemma 76}\end{aligned}$$

and since $im(f)$ is in $\omega\mathbf{RP}$, we have obtained the desired fixed point. If F is an endofunctor on the ω-profinite domains then the map d can be defined on the cpo of *algebraic* deflations so its fixed point will be algebraic and hence profinite. □

Chapter 6

Functor Continuity and Fixed Points

In this chapter we take a short look at several functors and we discuss the *categorical* technique for solving domain equations.

6.1 Fixed point existence and coproducts

Definition: Let $\mathbf{C}$ and $\mathbf{C}'$ be categories and $F : \mathbf{C} \to \mathbf{C}'$ a functor. If $\Delta = \langle A_i, \mu_{ij} \rangle_{i,j \in I}$ is an inverse system over $\mathbf{C}$ in order type I and $\mu : A \to \Delta$ is a limiting cone we set

$$F(\Delta) = \langle F(A_i), F(\mu_{ij}) \rangle_{i,j \in I} \text{ and}$$
$$F(\mu_i) = \langle F(\mu_i) \rangle_{i \in I}.$$

The functor F is said to be *continuous* if for every inverse system Δ and limiting cone $\mu : A \to \Delta$, the cone $F(\mu) : F(A) \to F(\Delta)$ is limiting. □

Theorem 78 *Suppose $F : (\mathbf{CPO}^P)^n \to \mathbf{CPO}^P$ is continuous. If $F(A_1, \ldots, A_n)$ is finite whenever $A_1, \ldots, A_n$ are finite then $F(D_1, \ldots, D_n)$ is profinite whenever $D_1, \ldots, D_n$ are profinite.*

Proof. To simplify the notation, assume that F is *binary.* The proof for an n-ary functor is essentially the same. Suppose D, E are profinite. Let $\langle A_i, a_{ij} \rangle_{i,j \in I}$ and $\langle B_i, b_{ij} \rangle_{i,j \in J}$ be inverse systems in $\mathbf{CPO}^P$ such that

- A_i is finite for each $i \in I$, and $D \cong \varprojlim \langle A_i, a_{ij} \rangle_{i,j \in I}$;
- B_i is finite for each $i \in J$, and $E \cong \varprojlim \langle B_i, b_{ij} \rangle_{i,j \in J}$.

Let $K = I \times J$. With the cordinatewise ordering, K is directed. For each $k = (i,j) \in K$, set $A'_k = A_i$ and $B'_k = B_j$. If $k = (i,j)$ and $l = (m,n)$ are in K and $k \geq l$ then set $a'_{kl} = a_{im}$ and $b'_{kl} = b_{jn}$. Now, set $C_k = (A'_k, B'_k)$ and $c_{kl} = (a'_{kl}, b'_{kl})$ for each $k \in K$. Then $\langle C_k, c_{kl} \rangle_{k,l \in K}$ is an inverse system in $\mathbf{CPO}^P \times \mathbf{CPO}^P$. We have

$$\begin{aligned} F(D,E) &\cong F(\varprojlim \langle A_i, a_{ij} \rangle_{i,j \in I}, \varprojlim \langle B_i, b_{ij} \rangle_{i,j \in J}) \\ &\cong F(\varprojlim \langle A'_k, a_{kl} \rangle_{k,l \in K}, \varprojlim \langle B'_k, b_{kl} \rangle_{k,l \in K}) \\ &\cong F(\varprojlim \langle C'_k, c_{kl} \rangle_{k,l \in K}) \\ &\cong \varprojlim \langle F(C_k), F(c_{kl}) \rangle_{k,l \in K}. \end{aligned}$$

But $F(C_k)$ is finite for each $k \in K$ since A_k and B_k are finite. Thus $F(D, E)$ is profinite. □

Theorem 79 *If $F : \mathbf{P}^P \to \mathbf{P}^P$ is a continuous functor then F has a profinite fixed point with root A if and only if there is a poset $A \cong \mathrm{rt}(F(A))$.*

Proof. Suppose D is profinite and $D \cong F(D)$. If A is the root of D then $A \cong \mathrm{rt}(F(D))$. Now, $F(A)$ is a projection of $F(D)$ by continuity so $\mathrm{rt}(F(A)) \cong \mathrm{rt}(F(D))$. Suppose, on the other hand that $A \cong \mathrm{rt}(F(A))$. Then there is a projection $p : F(A) \to A$ and this induces an inverse system,

$$A \stackrel{p}{\longleftarrow} F(A) \stackrel{F(p)}{\longleftarrow} F^2(A) \stackrel{F^2(p)}{\longleftarrow} \cdots$$

which has a profinite limit D. But

$$\begin{aligned} F(D) &= F(\varprojlim F^i(A)) \\ &\cong \varprojlim F^{i+1}(A) \\ &\cong D \end{aligned}$$

by continuity. □

Let A and B be posets having property m and suppose u, v are subsets of A and B respectively. Then Lemma 49 and an easy induction may be used to show that

$$\mathcal{U}^n_{A\times B}(u \times v) = \mathcal{U}^n_A \times \mathcal{U}^n_B(v)$$

for each $n \in \omega$. Hence, in particular, $\mathrm{rt}(A \times B) = \mathcal{U}^*_{A\times B}(\emptyset) = \mathcal{U}^*_A(\emptyset) \times \mathcal{U}^*_B(\emptyset) = \mathrm{rt}(A) \times \mathrm{rt}(B)$. Moreover, we have the following:

Theorem 80 *The product functor is continuous on* **CPO**.

Proof. Let $\langle D_i, d_{ij}\rangle$ and $\langle E_i, e_{ij}\rangle$ be **CPO** inverse systems of the same order type. Then

$$\langle D_i \times E_i, d_{ij} \times e_{ij}\rangle$$

is an inverse system. To show that

$$(\varprojlim \langle D_i, d_{ij}\rangle) \times (\varprojlim \langle E_i, e_{ij}\rangle) \cong \varprojlim \langle D_i \times E_i, d_{ij} \times e_{ij}\rangle.$$

one verifies that the functions

$$\begin{aligned} &f : D_* \times E_* \to F_* \text{ given by } f(x, y) = \langle x_i, y_i\rangle, \text{ and} \\ &g : F_* \to D_* \times E_* \text{ given by } g(\langle (x_i, y_i)\rangle) = (\langle x_i\rangle, \langle y_i\rangle) \end{aligned}$$

make sense and are inverse to one another. Since f and g are monotone we therefore have the desired isomorphism. □

In light of Theorem 79 this is noteworty in the following regard. Since the product is continuous, the functor $F(D) = D \times D$ is continuous and we know that it sends profinite domains to profinite domains. Suppose $D \cong F(D)$ is profinite and let $A = \mathrm{rt}(D)$. Now, A is finite so suppose it has m elements. Then $\mathrm{rt}(F(D)) = \mathrm{rt}(D \times D) = A \times A$ has m^2

elements. Since $\mathrm{rt}(D) \cong \mathrm{rt}(F(D))$ we must have $m = m^2$ so apparently $m = 1$ or $m = 0$. In other words, a non-empty profinite fixed point of the equation $D \cong F(D)$ must have a least element.

A similar fact holds for the functor $F(D) = \mathbf{CPO}(D, D)$. Suppose A is a non-empty finite poset and $A \cong \mathrm{rt}(A^A)$. We claim that A is the trivial one element poset. To see this, suppose A is non-trivial. Then A has a set of n minimal elements where $n > 1$. Now, a constant function mapping all of A to a minimal element of A is minimal in $\mathbf{CPO}(A, A)$ so $\mathrm{rt}(\mathbf{CPO}(A, A))$ has a least n minimal elements and none of these constant functions is equal to the identity function. Let $f : A \to A$ be monotone and suppose f is below the identity function on A. Suppose $X \in A$ and $f(Y) = Y$ for every $Y \sqsubset X$. Since A is simple there is a set $u \subseteq A$ such that $X \in \mathrm{MUB}(u)$. For if this were not the case then X could not lie in $\mathcal{U}^n(\emptyset)$ for any n. But then $u = f(u) \sqsubseteq f(X) \sqsubseteq X$ so $f(X) = X$. Hence f is the identity function and consequently the identity function is minimal in $\mathbf{CPO}(A, A)$. But this means $\mathrm{rt}(\mathbf{CPO}(A, A))$ has at least $n + 1$ minimal elements so we cannot have $A \cong \mathrm{rt}(\mathbf{CPO}(A, A))$. This shows that a non-empty profinite fixed point of the functor F must have a least element. These observations, together with Theorem 47 can be used to prove the following

Theorem 81 *If D is a non-empty ω-algebraic cpo and $D \cong \mathbf{CPO}(D, D)$ then D has a least element.* □

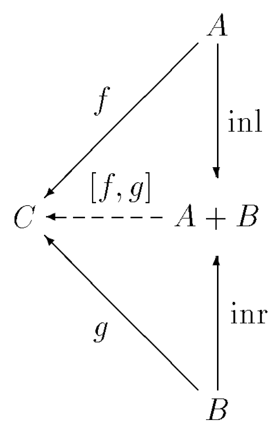
It is, incidently, *not true* in general that for a poset A, $\mathbf{CPO}(\mathrm{rt}(A), \mathrm{rt}(B)) \cong \mathrm{rt}(\mathbf{CPO}(A, B))$. Consider, for example, the opposite T^{op} of the truth value cpo. The monotone functions from T^{op} into T^{op} form a poset whose root is not isomorphic to the poset $\mathbf{CPO}(\mathrm{rt}(T^{op}), \mathrm{rt}(T^{op})) = \mathbf{CPO}(T^{op}, T^{op})$. Hasse diagrams for T^{op} and $\mathbf{CPO}(T^{op}, T^{op})$ appear in Figure 6.1 (see page 72). The root of $\mathbf{CPO}(T^{op}, T^{op})$ is drawn in black there.

These two examples illustrate a number of problems involved in obtaining profinite solutions to recursive domain equations. While Theorem 79 completely specifies which continuous endofunctors on $\mathbf{P}^P$ have fixed points it does not say that the fixed point will be non-trivial. Indeed, if the construction in the theorem is carried out for the functor $F(D) = D \times D$, the derived solution will be the one element domain 1.

Also, there are interesting and natural endo-functors on $\mathbf{P}$ for which there is *no* non-empty finite poset satisfying $A \cong \mathrm{rt}(F(A))$. One especially noteworthy example of this is the diagonal of the coproduct functor $+$. For arbitrary pre-orders A and B the coproduct is defined as follows. We let $A + B = (A \times \{0\}) \cup (B \times \{1\})$ and say $(X, n) \vdash_{A+B} (Y, m)$ if and only if either

- $n = m = 0$ and $X \vdash_A Y$, or
- $n = m = 1$ and $X \vdash_B Y$.

In essence, $A + B$ is the pre-order obtained by forming the disjoint union of A and B. This differs from the $+$ which appears in most of the literature on domain theory. The sum which appears in references such as [Stoy 1977], [Scott 1982a] or [Brookes 1984] is either the separated or coalesced sum and is *not* a categorical coproduct. A binary operation $+$ in a category is said to be a *coproduct* if for every pair of objects A, B, there are arrows inl and inr such that for every object C and pair f, g of arrows, there is a unique arrow $[f, g]$ which completes the following diagram



For pre-orders A and B, we define inl and inr by

$$\forall X \in A.\ X \text{ inl } (Y, 0) \text{ if and only if } X \vdash_A Y,$$
$$\forall X \in B.\ X \text{ inr } (Y, 1) \text{ if and only if } X \vdash_B Y$$

and given C, f and g as above we set

$$(X, 0)\ [f, g]\ Y \text{ if and only if } X\ f\ Y,$$
$$(X, 1)\ [f, g]\ Y \text{ if and only if } X\ g\ Y.$$

It follows easily from these definitions that $[f, g] \circ \text{inl} = f$ and $[f, g] \circ \text{inr} = g$. To see that $[f, g]$ is uniquely determined by these equations, suppose $h \circ \text{inl} = f$ for an approximable h. If $(X, 0)\ h\ Y$ for some $X \in A$, then $X \text{ inl } (X, 0)$ and $h \circ \text{inl} = f$ implies $X\ f\ Y$. Hence $(X, 0)\ h\ Y$ implies $(X, 0)\ [f, g]\ Y$. On the other hand, if $X \in A$ and $X\ f\ Z$ then $X\ h \circ \text{inl}\ Z$ so there is some $Y \in A$ such that $X \text{ inl } (Y, 0)$ and $(Y, 0)\ h\ Z$. But then $(X, 0) \vdash_{A+B} (Y, 0)$ so $(X, 0)\ h\ Z$ by the approximability of h. Hence $(X, 0)\ [f, g]\ Y$ implies $(X, 0)\ h\ Z$. If $h \circ \text{inr} = g$ then a similar pair of arguments shows that for any $(X, 1) \in B$ we have $(X, 1)\ h\ Y$ if and only if $(X, 1)\ [f, g]\ Y$. Hence, we must have $h = [f, g]$.

The coproduct $+$ on pre-orders induces a coproduct on algebraic cpo's. Indeed $|A| + |B| \cong |A + B|$ for any pair A, B of pre-orders. It is an endofunctor on the Plotkin orders and hence also on the profinite domains. But the diagonal functor $F(A) = A + A$ has only the empty poset as a fixed point. For if A is a Plotkin order then $\text{rt}(A+A) = \text{rt}(A) + \text{rt}(A)$ so $A \cong A + A$ implies $\text{rt}(A) \cong \text{rt}(A) + \text{rt}(A)$. But the only finite poset which can satisfy this is the empty one and the only Plotkin order with an empty root is the empty poset. This does not mean that this diagonal functor has no *cpo* as a fixed point. It is not difficult to check that $+$ is a continuous functor on **CPO** and the diagonal functor has many complete posets as fixed points. For example, any infinite discrete set will do. The moral is this: even a nice continuous endofunctor on **P** may not have non-trivial profinite fixed points.

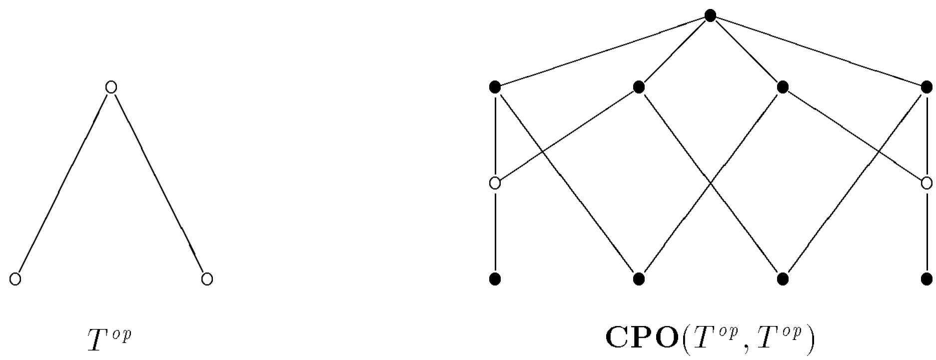


Figure 6.1: Root of a function space

6.2 Continuity of the exponential functor

Fortunately, almost all of the functors which one is inclined to use in denotational semantics are continuous. There do exist discontinuous, interesting functors such as ideal completion $|\cdot|$ but for the most part there is little cause to look for their fixed points except as a mathematical exercise. However, finding fixed points for equations involving the functor $\mathbf{CPO}(\cdot,\cdot)$ *is* important. But $\mathbf{CPO}(\cdot,\cdot)$ is not continuous! Recall that this functor is contravariant in its first argument. The problem is typically remedied by replacing $\mathbf{CPO}(\cdot,\cdot)$ by a new functor which is defined on $\mathbf{CPO}^P$ and is covariant in both of its arguments (see [Smyth and Plotkin 1982]). We show that this approach may be extended to yield a continuous functor on $\mathbf{CPO}^{\uparrow} \times \mathbf{CPO}^{\uparrow} \to \mathbf{CPO}^{\uparrow}$ as follows. On objects D, E we let $[D \to E] = \mathbf{CPO}(D, E)$. If D' and E' are cpo's and $\langle p, q\rangle : E \xrightarrow{\text{adj}} D$, $\langle p', q'\rangle : E' \xrightarrow{\text{adj}} D'$ are continuous then define

$$[p \to p'] : [E \to E'] \to [D \to D']$$

by $p \to p' : f \mapsto p' \circ f \circ q$. To see that this does indeed define a functor we begin by showing that the function $t : [D \to D'] \to [E \to E']$ by $t : f \mapsto q' \circ f \circ p$ is lower adjoint to $[p \to p']$. For if $f : E \to E'$, then

$$\begin{aligned}(t \circ [p \to p'])(f) &= t(p' \circ f \circ q)\\ &= (q' \circ p') \circ f \circ (q \circ p)\\ &\sqsubseteq \mathrm{id}_{E'} \circ f \circ \mathrm{id}_E\\ &= f.\end{aligned}$$

On the other hand, if $f : D \to D'$ then

$$\begin{aligned}([p \to p'] \circ t)(f) &= [p \to p'](q' \circ f \circ p)\\ &= (p' \circ q') \circ f \circ (p \circ q)\\ &\sqsupseteq \mathrm{id}_{D'} \circ f \circ \mathrm{id}_D\\ &= f.\end{aligned}$$

Now, if $\langle r, s\rangle : F \xrightarrow{\text{adj}} E$, $\langle r', s'\rangle : F' \xrightarrow{\text{adj}} E'$ and $f : F \to F'$ then

$$\begin{aligned}[p \to p'] \circ [r \to r'](f) &= [p \to p'](r' \circ f \circ s) \\ &= (p' \circ r') \circ f \circ (s \circ q) \\ &= [(p \circ r) \to (p' \circ r')](f)\end{aligned}$$

since $s \circ q$ is the lower adjoint corresponding to $p \circ r$.

Theorem 82 *The functor $[\cdot \to \cdot]$ is continuous in* $\mathbf{CPO}^{\uparrow}$.

Proof. Let $\langle D_i, d_{ij}\rangle$ and $\langle E_i, e_{ij}\rangle$ be inverse systems of order type I in $\mathbf{CPO}^{\uparrow}$. For each $i \geq j$ define $f_{ij} : [D_i \to E_i] \to [D_j \to E_j]$ by $f_{ij} : f \mapsto e_{ji} \circ f \circ d_{ij}$. Then we must show that

$$[(\varprojlim \langle D_i, d_{ij}\rangle) \to (\varprojlim \langle E_i, e_{ij}\rangle)] \cong \varprojlim \langle [D_i \to E_i], f_{ij}\rangle.$$

If $i \geq j$, define $f_{ji} : [D_j \to E_j] \to [D_i \to E_i]$ by $f_{ji} : \phi \mapsto e_{ji} \circ \phi \circ d_{ji}$. We have already shown that $\langle f_{ij}, f_{ji}\rangle : [D_i \to E_i] \xrightarrow{\text{adj}} [D_j \to E_j]$ and $f_{jk} \circ f_{ij} = f_{ik}$ if $i \geq j \geq k$. To simplify the notation, let $F_* = \varprojlim(\langle [D_i \to E_i], f_{ij}\rangle)$. Now, suppose $\phi \in F_*$. We wish to define a function $\phi^\sharp : D_* \to E_*$ by the equations

$$e_{*i} \circ \phi^\sharp = \bigsqcup_{j \geq i} e_{ji} \circ \phi_j \circ d_{*j},\ i \in I.$$

To see that the set on the right is directed, suppose $k \geq j \geq i$. Then

$$\begin{aligned}e_{ji} \circ \phi_j \circ d_{*j} &= e_{ji} \circ (e_{kj} \circ \phi_k \circ d_{kj}) \circ d_{*j} \\ &= (e_{ji} \circ e_{kj}) \circ \phi_k \circ (d_{jk} \circ d_{*j}) \\ &= e_{ki} \circ \phi_k \circ (d_{jk} \circ d_{kj} \circ d_{*k}) \\ &\sqsubseteq e_{ki} \circ \phi_k \circ d_{*k}\end{aligned}$$

so the set is directed since I is. To see that this sup really does yield an element of E_*, suppose $j \geq i$. then

$$\begin{aligned}e_{ji} \circ (e_{*j} \circ \phi^\sharp) &= e_{ji} \circ (\bigsqcup_{k \geq j} e_{kj} \circ \phi_k \circ d_{*k}) \\ &= \bigsqcup_{k \geq j} (e_{ji} \circ e_{kj}) \circ \phi_k \circ d_{*k} \\ &= \bigsqcup_{k \geq i} e_{ki} \circ \phi_k \circ d_{*k} \\ &= e_{*i} \circ \phi^\sharp.\end{aligned}$$

The continuity of $\phi^\sharp$ is obvious.

Now suppose on the other hand that $\phi : D_* \to E_*$ is continuous. Then ϕ defines a unique sequence $\phi^\flat = \langle e_{*i} \circ \phi \circ d_{i*}\rangle$. We claim that $\phi^\flat \in C_*$. For if $i \geq j$ then

$$\begin{aligned}f_{ij}(\phi^\flat_i) &= e_{ij} \circ (e_{*i} \circ \phi \circ d_{i*}) \circ d_{ji} \\ &= (e_{ij} \circ e_{*i}) \circ \phi \circ (d_{i*} \circ d_{ji}) \\ &= e_{*j} \circ \phi \circ d_{j*} \\ &= \phi^\flat_j.\end{aligned}$$

We now demonstrate that the maps $\phi \mapsto \phi^\sharp$ and $\phi \mapsto \phi^\flat$ are inverse to one another. Taking the easy case first, suppose $\phi : D_* \to E_*$ is continuous.

$$\begin{aligned} e_{*i} \circ (\phi^{\flat\sharp}) &= \bigsqcup_{j \geq i} e_{ji} \circ \phi^\flat_j \circ d_{*j} \\ &= \bigsqcup_{j \geq i} e_{ji} \circ (e_{*j} \circ \phi \circ d_{j*}) \circ d_{*j} \\ &= \bigsqcup_{j \geq i} (e_{ji} \circ e_{*j}) \circ \phi \circ (d_{j*} \circ d_{*j}) \\ &= e_{*i} \circ \phi \end{aligned}$$

by Lemma 29. Before starting the other direction the reader is advised to sit down. Suppose $\phi \in F_*$ and $i \in I$. Then

$$\begin{aligned} e_{*i} \circ \phi^\sharp \circ d_{i*} &= (\bigsqcup_{j \geq i} e_{ji} \circ \phi_j \circ d_{*j}) \circ d_{i*} \\ &= \bigsqcup_{j \geq i} e_{ji} \circ \phi_j \circ (d_{*j} \circ d_{i*}) \\ &= \bigsqcup_{j \geq i} e_{ji} \circ \phi_j \circ (\bigsqcup_{k \geq i,j} d_{kj} \circ d_{ik}) && \text{by definition of } d_{i*} \\ &= \bigsqcup_{k \geq j \geq i} e_{ji} \circ \phi_j \circ d_{kj} \circ d_{ik} \\ &= \bigsqcup_{k \geq j \geq i} e_{ji} \circ (e_{kj} \circ \phi_k \circ d_{jk}) \circ d_{kj} \circ d_{ik} && f_{kj}(\phi_k) = \phi_j \\ &= \bigsqcup_{k \geq j \geq i} (e_{ji} \circ e_{kj}) \circ \phi_k \circ (d_{jk} \circ d_{kj} \circ d_{ik}) \\ &= \bigsqcup_{k \geq j \geq i} e_{ki} \circ \phi_k \circ (d_{jk} \circ d_{kj} \circ d_{jk}) \circ d_{ij} && k \geq j \geq i \\ &= \bigsqcup_{k \geq j \geq i} e_{ki} \circ \phi_k \circ (d_{jk} \circ d_{ij}) && \text{Lemma 17} \\ &= \bigsqcup_{k \geq j \geq i} e_{ki} \circ \phi_k \circ d_{ik} \\ &= \phi_i && f_{ki}(\phi_k) = \phi_i. \end{aligned}$$

Since the maps $\phi \mapsto \phi^\sharp$ and $\phi \mapsto \phi^\flat$ are monotone, we have the desired isomorphism. □

6.3 Powerdomains and other functors

In this section we define powerdomains and show how the methods that have been introduced can be used to solve a domain equation up to *equality* rather than just isomorphism. We also discuss a couple of other noteworthy functors: the lifting functor and the join completion functor.

Definition: Let A be a pre-order and suppose M is the set of finite subsets of A. The *upper powerdomain* $\mathcal{Q}(A)$ of A is the set M together with a pre-ordering $\vdash_{\mathcal{Q}(A)}$ given by

$$u \vdash_{\mathcal{Q}(A)} v \text{ if } (\forall X \in u)(\exists Y \in v).\ X \vdash_A Y.$$

Dually, the *lower powerdomain* $\mathcal{R}(A)$ of A is M with the pre-ordering $\vdash_{\mathcal{R}(A)}$ given by

$$u \vdash_{\mathcal{R}(A)} v \text{ if } (\exists Y \in v)(\forall X \in u).\ X \vdash_A Y.$$

The *convex powerdomain* $\mathcal{S}(A)$ of A is the intersection of the upper and lower powerdomain pre-orderings on M, *i.e.*

$$u \vdash_{\mathcal{S}(A)} v \text{ if } u \vdash_{\mathcal{Q}(A)} v \text{ and } u \vdash_{\mathcal{R}(A)} v.$$

If $f : A \to B$ is approximable then the action of $\mathcal{Q}, \mathcal{R}, \mathcal{S}$ on f is given by

$$\begin{array}{l} u\ \mathcal{Q}(f)\ v \text{ if } (\forall X \in u)(\exists Y \in v).\ X\ f\ Y \\ u\ \mathcal{R}(A)\ v \text{ if } (\exists Y \in v)(\forall X \in u).\ X\ f\ Y \\ u\ \mathcal{S}(A)\ v \text{ if } u\ \mathcal{Q}(f)\ v \text{ and } u\ \mathcal{R}(f)\ v. \end{array}$$

□

The *lifting* operation $F(A) = A_\perp$ on pre-orders A is defined as follows. For simplicity, assume that $\perp$ is a new element that is not in A. Then

- $A_\perp = A \cup \{\perp\}$,
- $X \vdash_{A_\perp} \perp$ for each $X \in A_\perp$,
- $X \vdash_{A_\perp} Y$ for $X, Y \in A$ if $X \vdash_A Y$.

Lemma 83 *A poset D is algebraic if and only if $D_\perp$ is algebraic.*

Proof. We claim that for any pre-order A, $|A|_\perp \cong |A_\perp|$. To see this, define $f : |A|_\perp \to |A_\perp|$ by

$$f(x) = \begin{cases} \{\perp\} & \text{if } x = \perp; \\ x \cup \{\perp\} & \text{otherwise.} \end{cases}$$

and define $g : |A_\perp| \to |A|_\perp$ by

$$g(x) = \begin{cases} \perp & \text{if } x = \{\perp\}; \\ x - \{\perp\} & \text{otherwise.} \end{cases}$$

The functions f, g are obviously monotone and the proof that $f \circ g = \mathrm{id}_{|A_\perp|}$ and $g \circ f = \mathrm{id}_{|A|_\perp}$ is a routine verification of cases. Now, if D is algebraic then $D \cong |\mathbf{B}[D]|$ so

$$D_\perp \cong |\mathbf{B}[D]|_\perp \cong |\mathbf{B}[D]_\perp| = |\mathbf{B}[D_\perp]|.$$

Hence $D_\perp$ is algebraic. If on the other hand, $D_\perp$ is algebraic then

$$D_\perp \cong |\mathbf{B}[D_\perp]| = |\mathbf{B}[D]_\perp| \cong |\mathbf{B}[D]|_\perp$$

so $D \cong |\mathbf{B}[D]|$ and D is therefore algebraic. □

On **CPO**, $(\cdot)_\perp$ can be made a functor by letting $f_\perp : D_\perp \to E_\perp$ by $f_\perp(x) = f(x)$ if $x \in D$ and $\perp$ otherwise. It is easy to show that $(\cdot)_\perp$ is continuous on $\mathbf{CPO}^P$. Hence, by Theorem 78, $D_\perp$ is profinite if D is. The converse is false, however. For example, the infinite discrete set $\mathcal{N}$ is not profinite (because it has an infinite root), but $\mathcal{N}_\perp$ is profinite.

Assume P is a pre-order none of whose members are pairs. We may define a pre-order $(A, \vdash)$ which satisfies the equation $A \cong (P + \mathcal{S}(A))_\perp$ as follows:

- $\perp \in A$,
- if $X \in P$ then $(X, 0) \in A$,
- if $u \subseteq A$ is finite then $(u, 1) \in A$,
- if $X \vdash \perp$ for every $X \in A$,
- $(X, 0) \vdash (Y, 0)$ if $X \vdash_P Y$,
- $(u, 1) \vdash (v, 1)$ if $u \vdash_{\mathcal{S}(A)} v$.

Proposition 84 *Let D be a cpo with property m. Then D is bounded complete if and only if D has greatest lower bounds for non-empty subsets.*

Proof. Suppose D is bounded complete and $S \subseteq D$. Then $M = \{\downarrow x \mid x \in S\}$ is directed so $\bigsqcup M$ exists and is the greatest lower bound of S. Suppose on the other hand that D has greatest lower bounds for non-empty subsets and $u \subseteq D$ is finite bounded set. Let x be the greatest lower bound for MUB(u). Then x is a bound for u. Since D has property m we must have $x \sqsupseteq y$ for some $y \in$ MUB(u). But this means $x \in$ MUB(u) and this is only possible if $u = \{x\}$. Hence x is a least upper bound for u. □

Definition: For a pre-order $\langle A, \vdash \rangle$ define the *join completion*

$$\langle \mathcal{J}(A), \vdash_{\mathcal{J}(A)} \rangle$$

as follows

- $\mathcal{J}(A) = \{u \subseteq A \mid u \text{ is finite and bounded}\}$
- $u \vdash_{\mathcal{J}(A)} v$ if and only if $\forall X \in A.\ X \vdash u \Rightarrow X \vdash v$. □

Theorem 85 *Let A and B be pre-orders. Then*

1. $\langle \mathcal{J}(A), \vdash_{\mathcal{J}(A)} \rangle$ *is bounded complete;*
2. *if A is bounded complete then $\mathcal{J}(A) \cong A$;*
3. *if $A \triangleleft B$ then $\mathcal{J}(A) \triangleleft \mathcal{J}(B)$;*
4. $\mathcal{J}(A \times B) \cong \mathcal{J}(A) \times \mathcal{J}(B)$.

Proof. (1) Suppose $u, v \in \mathcal{J}(A)$ and $w \vdash_{\mathcal{J}(A)} u, v$. Then $u \cup v$ is bounded in A by anything that bounds w. Hence $u \cup v$ is in $\mathcal{J}(A)$ and $w \vdash_{\mathcal{J}(A)} u \cup v$. But any bound for $u \cup v$ in A is a bound for u and a bound for v, so $u \cup v \vdash_{\mathcal{J}(A)} u, v$. Thus $\mathcal{J}(A)$ has bounded joins.

(2) Suppose A is bounded complete and define $f \subseteq A \times \mathcal{J}(A)$ by $X\ f\ u$ if and only if $X \vdash_A u$. To see that f is approximable, just note that $X\ f\ u$ if and only if $X \vdash_A Y$ where Y is a least upper bound for u. Hence, if $X\ f\ u, v$ then $X \vdash_A Y$ where Y is the least upper bound of $u \cup v$ so $X\ f\ u \cup v \vdash_{\mathcal{J}(A)} u, v$. The other conditions for approximability of f are obviously satisfied. Define $g \subseteq \mathcal{J}(A) \times A$ by $u\ g\ X$ if and only if $u \vdash_{\mathcal{J}(A)} \{X\}$ If $u\ g\ X$

and $u\ g\ Y$ then $u\ g\ Z$ where Z is a least upper bound for u. The remaining condition for approximability of g is obviously satisfied. Now suppose $X\ f\ u$ and $u\ g\ Z$ for some $X, Z \in A$ and $u \in \mathcal{J}(A)$. If Y is a least upper bound for u then $X \vdash_A Y \vdash_A Z$ so $X \vdash_A Z$. Therefore $g \circ f \subseteq \mathrm{id}_A$. If, on the other hand, $X\ (g \circ f)\ Z$ then there is a u such that $X\ f\ u$ and $u\ g\ Z$. If Y is a least upper bound for u then $X \vdash_A Y \vdash_A Z$. Hence $g \circ f = \mathrm{id}_A$. Now, suppose $u\ g\ X$ and $X\ w$ for some $u, w \in \mathcal{J}(A)$ and $X \in A$. Then $u \vdash_{\mathcal{J}(A)} \{X\}$ and $X \vdash_A Y$ where Y is a least upper bound of w. Hence $\{X\} \vdash_{\mathcal{J}(A)} \{Y\} \vdash_{\mathcal{J}(A)} w$ so $u \vdash_{\mathcal{J}(A)} w$. Therefore $f \circ g \subseteq \mathrm{id}_{\mathcal{J}(A)}$. If, on the other hand, $u \vdash_{\mathcal{J}(A)} w$ then $u \vdash_{\mathcal{J}(A)} \{Y\}$ for a least upper bound Y of w so $u\ g\ Y$ and $Y\ f\ w$. Hence $f \circ g = \mathrm{id}_{\mathcal{J}(A)}$.

(3) Suppose $A \lhd B$. If u is bounded in A then it is bounded in B so any element of $\mathcal{J}(A)$ is also an element of $\mathcal{J}(B)$. Suppose $u, v \in \mathcal{J}(A)$ and $u \vdash_{\mathcal{J}(A)} v$. We claim that $u \vdash_{\mathcal{J}(B)} v$. Suppose $X \in B$ and $X \vdash_A u$. Since $A \lhd B$, there is an $X' \in A$ such that $X \vdash_A X' \vdash_A u$. But $u \vdash_{\mathcal{J}(A)} v$ means $X' \vdash_A v$. Hence $X \vdash_A v$ and the claim is established. Obviously, $u \vdash_{\mathcal{J}(B)} v$ implies $u \vdash_{\mathcal{J}(A)} v$. Thus $\langle \mathcal{J}(A), \vdash_{\mathcal{J}(A)}\rangle \subseteq \langle \mathcal{J}(B), \vdash_{\mathcal{J}(B)}\rangle$. To see that $\mathcal{J}(A) \lhd \mathcal{J}(B)$, suppose $u, v \in \mathcal{J}(A)$ and $w \vdash_{\mathcal{J}(B)} u, v$ for some $w \in \mathcal{J}(B)$. If $X \vdash_A w$ for some $X \in B$ then $X \vdash_A u \cup v$ so $u \cup v$ is bounded and there is an $X' \in A$ such that $X' \vdash_A u \cup v$. Hence $u \cup v \in \mathcal{J}(A)$ and we conclude that $\mathcal{J}(A)$ is closed under existing joins in $\mathcal{J}(B)$. Thus $\mathcal{J}(A) \lhd \mathcal{J}(B)$.

(4) Define a relation $f : \mathcal{J}(A) \times \mathcal{J}(B) \to \mathcal{J}(A \times B)$ by $(u, v)\ f\ w$ iff $u \vdash_{\mathcal{J}(A)} \mathrm{fst}(w)$ and $v \vdash_{\mathcal{J}(B)} \mathrm{snd}(w)$. Define another relation $g : \mathcal{J}(A \times B) \to \mathcal{J}(A) \times \mathcal{J}(B)$ by $w\ g\ (u, v)$ iff $\mathrm{fst}(w) \vdash_{\mathcal{J}(A)} u$ and $\mathrm{snd}(w) \vdash_{\mathcal{J}(B)} u$. We claim that f and g are approximable. Starting with f, suppose $(u, v) \in \mathcal{J}(A) \times \mathcal{J}(B)$. Then $u \times v \in \mathcal{J}(A \times B)$ and $(u, v)\ f\ (u \times v)$. If $(u, v)\ f\ w$ and $(u, v)\ f\ w'$ then $\mathrm{fst}(w) \cup \mathrm{fst}(w') = \mathrm{fst}(w \cup w')$ and $\mathrm{snd}(w) \cup \mathrm{snd}(w') = \mathrm{snd}(w \cup w')$ are less than u and v respectively. Thus $(u, v)\ f\ (w \cup w')$. The remaining condition for approximability of f is obviously satisfied. To see that g is approximable, suppose $w\ g\ (u, v)$ and $w\ g\ (u', v')$. Then $\mathrm{fst}(w) \vdash_{\mathcal{J}(A)} u \cup u'$ and $\mathrm{snd}(w) \vdash_{\mathcal{J}(B)} v \cup v'$ so $w\ g\ (u \cup u', v \cup v') \vdash_{\mathcal{J}(A)\times\mathcal{J}(B)} (u, v)$. The other conditions are also straight-forward. Now, if $w\ (f \circ g)\ w'$ then $w\ g\ (u, v)\ f\ w'$ for some (u, v) so $\mathrm{fst}(w) \vdash_{\mathcal{J}(A)} \mathrm{fst}(w')$ and $\mathrm{snd}(w) \vdash_{\mathcal{J}(B)} \mathrm{snd}(w')$. Hence $w \vdash_{\mathcal{J}(A\times B)} w'$ and we therefore have $f \circ g \subseteq \mathrm{id}_{\mathcal{J}(A\times B)}$. On the other hand, if $w \vdash_{\mathcal{J}(A\times B)} w'$ then $w\ g\ (\mathrm{fst}(w), \mathrm{snd}(w))\ f\ w'$. We conclude that $f \circ g = \mathrm{id}_{\mathcal{J}(A\times B)}$. If $(u, v)\ (g \circ f)\ (u', v')$ then $(u, v)\ f\ w\ g\ (u', v')$ for some w so $u \vdash_{\mathcal{J}(A)} \mathrm{fst}(w) \vdash_{\mathcal{J}(A)} u'$ and $v \vdash_{\mathcal{J}(B)} \mathrm{snd}(w) \vdash_{\mathcal{J}(B)} v'$. Hence $(u, v) \vdash_{\mathcal{J}(A)\times\mathcal{J}(B)} (u', v')$ and we have $g \circ f \subseteq \mathrm{id}_{\mathcal{J}(A)\times\mathcal{J}(B)}$. On the other hand, if $u \vdash_{\mathcal{J}(A)} u'$ and $v \vdash_{\mathcal{J}(B)} v'$ then $(u, v)\ f\ (u \times v)\ g\ (u', v')$. Thus $g \circ f = \mathrm{id}_{\mathcal{J}(A)\times\mathcal{J}(B)}$. Hence f defines the desired isomorphism. □

By Corollary 71, there is a Plotkin order $1^\star$ such that whenever A is a Plotkin order with a least element, we have $A \trianglelefteq 1^\star$. We may extract from Theorem 85 the following

Corollary 86 *If A is a bounded complete pre-order then $A \trianglelefteq \mathcal{J}(1^\star)$.*

Proof. Since A has a least element we know that $A \cong A'$ for some $A' \lhd \mathcal{J}(1^\star)$. But A' is bounded complete so $A' \cong \mathcal{J}(A')$. Hence $A \cong A \trianglelefteq \mathcal{J}(1^\star)$. □

Now, suppose u and v are finite bounded subsets of $1^\star$ such that $u, v \neq \{\bot\}$. Consider the diagram type

$$\Gamma(\mathbf{v}) = \{\bot \neq v\} \cup \{\mathbf{v} \sqsubseteq \mathbf{X} \mid X \in u \cup v\}.$$

This type is normal over $\mathcal{U}^*_{1^\star}(u \cup v)$ so it has a realization Z in $1^\star$. But $u \vdash_{\mathcal{J}(1^\star)} \{Z\}$, $v \vdash_{\mathcal{J}(1^\star)} \{Z\}$ and $\{Z\} \not\sim \{\bot\}$. This shows that no pair $u, v \neq \{\bot\}$ of bounded subsets of $\mathcal{J}(1^\star)$ can be complementary to one another. Hence $\mathcal{J}(1^\star)$ cannot be isomorphic to the countable atomless boolean algebra with its top element removed. We conclude that although $|\mathcal{J}(1^\star)|$ is projection universal for bounded complete algebraic cpo's, it is not isomorphic to Scott's universal domain $\mathcal{U}$.

Chapter 7

Partial Functions

Sometimes it is more natural to think of functions as *partial* rather than *total*. This may be simply because we do not wish to burden ourselves with the need to provide some arbitrary definition of the function on places outside its natural domain. For example, in the elementary calculus, we frequently think of functions such as $f(x) = 1/x$ as partially defined on the real numbers. In many instances the undefined points (singularities) are a primary topic of interest. In the theory of complex variables isolated undefined points of a meromorphism are classified as *removable singularities* or as *poles of finite order*. Other complex functions have what are called *essential singularities*. Of, course, we may think of a meromorphism as a total function on a certain kind of subset of the complexes. The fact is, however, that we think of such functions as living on a piece of the complex plane and consider the places where the function is *not* defined to be a significant object of attention.

Another area in which the use of partial functions is pervasive is recursive function theory. A recognition of the importance of *partial* recursive functions goes back to the inception of the subject and the reasons for considering partiality in recursive function theory are quite compelling. One cannot, in a uniformly effective way, tell whether an algorithm will converge on a given value and one cannot enumerate the Gödel numbers of the total functions. Rogers [1967] makes the following comments about this problem:

> Of course, situations may then arise where there is no evident way to determine whether a set of instructions yields a total function or not. Assume, for example, that we have an expression ... which embodies the instructions: "To compute $f(x)$, carry out the decimal expansion of π until a run of at least x consecutive 5's appears; if and when this occurs, give the position of the first digit of this run as output." Or, for a simpler example, take: "To compute $g(x)$, examine successive even numbers greater than 2 until one appears which is not the sum of two primes; if and when this occurs, give the output $g(x) = 0$." In each example ... we have a specific computing procedure but do not know whether this procedure gives a function, *i.e.* whether it always terminates and yields an output. What we *can* conclude is that each procedure gives a *partial function.* If it should happen to be true that there are runs of eight 5's but none of greater length in π, then the first example would give a set of instructions for a partial function whose domain consisted of the first nine integers. If Goldbach's conjecture is true, then

the second example would give the empty partial function; if the conjecture is false, then the second example would give the constant function $\lambda x[0]$. In any case, each example provides specific calculating instructions which determine a specific partial function.

It is quite natural therefore that we allow our functions to be undefined on some values and this is the course followed by Rogers and most other writers on the subject of recursive functions.

7.1 Partial functions on cpo's

Although partial recursive functions are defined on the natural numbers $\mathcal{N}$, we have perfectly good notions of computability for *higher types* as well. It will not be our purpose to discuss computability theory at higher types in this chapter, but the possiblity of getting such a theory does suggest that the notion of partiality higher types may be worthy of investigation. Starting with the category of *sets* we can get a perfectly good notion of partiality by taking functions which are defined on subsets. That is, a partial function $\phi : A \rightharpoonup B$ is a function $f : A' \to B$ where $A' \subseteq A$. There is a subtlety about types here since f is a perfectly good partial function in its own right. We refer the reader to sources such as Heller [1985] and Rosolini [1984] for more precise formulations. Given a set like $\mathcal{N}$, there is an associated set $[\mathcal{N} \rightharpoonup \mathcal{N}]$ of partial functions on $\mathcal{N}$. There is, however, a natural *order structure* on this set: given partial functions, $\phi : \mathcal{N} \rightharpoonup \mathcal{N}$ and $\psi : \mathcal{N} \rightharpoonup \mathcal{N}$ we can say $\phi \sqsubseteq \psi$ if whenever $x \in \mathcal{N}$ and ϕ is defined on x then ψ is defined on x and $\phi(x) = \psi(x)$. This suggests that we move to a more structured category of spaces. Since this order on $\mathcal{N}$ is complete and **CPO** has the sets as a subcategory, defining a notion of partial function on cpo's seems like a reasonable level at which to begin discussing higher type partial functions. In this section we establish some basic definitions and properties of cpo's with *continuous* partial functions defined on a Scott open sets.

Let D and E be cpo's and suppose $\phi : D \rightharpoonup E$ is a partial function. Let $dom(\phi)$ be the subset of D on which ϕ is defined. Write $\phi(x)\downarrow$ to indicate that $x \in dom(\phi)$ and in this case let $\phi(x)$ denote the value of ϕ on x. If $x \notin dom(\phi)$ then we write $\phi(x)\uparrow$ (and in this case the expression "$\phi(x)$" is non-denoting). An equation like $\phi(x) = y$ means that $\phi(x)\downarrow$ and y is the value of ϕ at x. We write $s \simeq t$ for terms s and t to mean that if s or t is defined then both are defined and $s = t$. A partial function ϕ is continuous if and only if $dom(\phi)$ is open and the restriction of ϕ to $dom(\phi)$ is continuous (as a total function). The following lemma is immediate from Lemma 1.

Lemma 87 *Suppose $\phi : D \rightharpoonup E$ is a partial function between cpo's D and E. Then ϕ is continuous if and only if for every directed $M \subseteq dom(\phi)$, $\phi(\bigsqcup M) = \bigsqcup \phi(M)$.* □

If $\phi, \psi : D \rightharpoonup E$ are partial functions then we write $\phi \sqsubseteq \psi$ if $dom(\phi) \subseteq dom(\psi)$ and $\phi(x) \sqsubseteq \psi(x)$ for each $x \in dom(\phi)$. For partial functions $\phi : D \rightharpoonup E$ and $\psi : E \rightharpoonup F$ we define $\psi \circ \phi : D \rightharpoonup F$ by

$$\psi \circ \phi \simeq \begin{cases} \psi(\phi(x)) & \text{if } \phi(x)\downarrow \text{ and } \psi(\phi(x))\downarrow; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

With the identity function for **CPO** and this composition the complete posets and partial continuous functions form a category which we call $\mathbf{CPO}_\partial$ (read as "cpo partial"). If **C** is a full subcategory of **CPO** then we write $\mathbf{C}_\partial$ for the category with the same objects as **C** and with partial continuous functions as arrows. Let $[D \rightharpoonup E]$ be the partially ordered set of continuous partial functions from D into E. There is a close relationship between partial functions and *strict* continuous functions. A (total) continuous function $f : D_\perp \to E_\perp$ is strict if $f(\perp_D) = \perp_E$. Specifically, we have the following

Lemma 88 *For any pair D, E of cpo's, $[D \rightharpoonup E] \cong [D \to E_\perp]$. In fact there is an isomorphism between $\mathbf{CPO}_\partial$ and the category $\mathbf{CPO}_\perp$ of cpo's with bottoms and strict functions.*

Proof. First, suppose $\phi : D \rightharpoonup E$ is partial continuous. Define $total_\phi : D \to E_\perp$ by

$$total_\phi = \begin{cases} \phi(x) & \text{if } \phi(x)\downarrow; \\ \perp & \text{otherwise.} \end{cases}$$

to see that this function is continuous, suppose $M \subseteq D$ is directed. If $M \cap dom(\phi) = \emptyset$ then $\bigsqcup M \notin dom(\phi)$ since $dom(\phi)$ is open so $total_\phi(\bigsqcup M) = \perp = \bigsqcup total_\phi(M)$. If $N = M \cap dom(\phi) \neq \emptyset$ then N is directed and $\bigsqcup M = \bigsqcup N$ so

$$\begin{aligned} total_\phi(\bigsqcup M) &= total_\phi(\bigsqcup N) \\ &= \phi(\bigsqcup N) \\ &= \bigsqcup \phi(N) \\ &= \bigsqcup total_\phi(M). \end{aligned}$$

It is clear also that if $\phi \sqsubseteq \psi$ then $total_\phi \sqsubseteq total_\psi$. Now, if $f : D \to E_\perp$ is continuous then let $f_\partial : D \rightharpoonup E$ be given by

$$f_\partial(x) \simeq \begin{cases} f(x) & \text{if } f(x) \neq \perp; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Since f is continuous and $E - \{\perp\}$ is open, $dom(f_\partial) = f^{-1}(E - \{\perp\})$ is also open. Hence the continuity of f_∂ is immediate. It is also clear that if $f \sqsubseteq g$ then $f_\partial \sqsubseteq g_\partial$. The maps *partial* and *total* are inverse to one another so they yield the desired isomorphism. The isomorphism between $\mathbf{CPO}_\partial$ and $\mathbf{CPO}_\perp$ comes from the fact that $[D \to E_\perp]$ is isomorphic to the poset of strict continuous functions from $D_\perp$ into $E_\perp$. More explicitly, if $\phi : D \rightharpoonup E$ then define $\phi_\perp : D_\perp \to E_\perp$ by

$$\phi_\perp(x) = \begin{cases} total_\phi(x) & \text{if } x \neq \perp; \\ \perp & \text{otherwise.} \end{cases}$$

Then the correspondences functors $(\cdot)_\perp$ and $(\cdot)_\partial$ define an isomorphism between $\mathbf{CPO}_\partial$ and $\mathbf{CPO}_\perp$. □

Certainly, there are a great many strict continuous functions between a pair of cpo's D, E having bottoms. For example, any surjection $p : E \to D$ is strict. Hence, in particular, projections are strict. Lower adjoints, since they preserve roots will also be strict. However, in general, an upper adjoint may not be strict.

There is a theory of categories with partial maps which captures in a rather general framework properties of partial functions over categories like sets and enumerated sets. Notions such as that of a *dominical* category ([Heller 1985], [Rosolini 1984]) or *partial* cartesian closed category [Longo and Moggi 1984] give a nice framework within which we can understand the most essential properties of partial functions. It is our contention that this applies to the categories of partial maps which we discuss in this chapter. Rather than develop this general theory here we simply cite some evidently categorical properties of our spaces and make some comments on how they support the claim that our choices of objects and morphisms are natural. The following lemma says (among other things) that the total functions on $\mathbf{CPO}_\partial$ can be distinguished categorically in terms of the totally undefined functions.

Lemma 89 *For each pair D, E of cpo's, let $\vartheta_{DE} : D \rightharpoonup E$ be the totally undefined map.*

1. *If $\phi : D \rightharpoonup E$ and F is a cpo then $\phi \circ \vartheta_{FD} = \vartheta_{FE}$ and $\vartheta_{EF} \circ \phi = \vartheta_{DF}$.*
2. *Moreover, ϕ is total if and only if for every cpo F and $\psi : F \rightharpoonup D$, $\phi \circ \psi = \vartheta_{FE}$ implies $\psi = \vartheta_{FD}$.*
3. *In $\mathbf{CPO}_\partial$, the empty set is both the initial and terminal object.* □

The category $\mathbf{CPO}_\perp$ represents a different "philosophy" of partiality from $\mathbf{CPO}_\partial$. With sets, for example, instead of taking functions which are truely partial we can take as objects sets with a distinguished point $*$ and take as arrows total functions which send $*$ to $*$. A total function $f : A \to B$ is then "undefined" on an argument x if $f(x) = *$. This category is equivalent to sets with partial functions. Note, however, that our earlier story about the passage from sets with partial functions to cpo's with partial functions does mean much with respect to these pointed sets and $\mathbf{CPO}_\perp$ since the sets are not a subcategory of $\mathbf{CPO}_\perp$. Although $\mathbf{CPO}_\perp$ and $\mathbf{CPO}_\partial$ are equivalent, it does seem that bottomless complete posets with continuous partial functions are a more naturally motivated class than cpo's with strict (total) functions. In the next section we also put forth the view that the functors of interest are also more elegantly represented when we leave off the bottoms and consider partial continuous functions.

7.2 Partial Plotkin orders and pre-domains

The category $\mathbf{CPO}_\partial$ is really very large for the purposes of domain theory. Although the objects of the category are easy to present through a simple axiomatization, their structure can be more complex than we would like to allow in general. In particular there seems to be no way to get a theory of computability on cpo's. What is needed is some sort of countable basis for the poset. Then one can view the computable elements in terms of approximations. The idea of continuity of a poset is perfectly suited to this purpose and work on this kind of computability at higher types abounds (as mentioned earlier [Weihrauch and Deil 1980] provides a pleasing intuitive introduction).[1] A more tractable class than the continuous

[1] Unfortunately the encyclopedic **Compendium of Continuous Lattices** [Gierz, *et. al.* 1981] has no discussion of the subject other than history.

posets is the *algebraic* posets. This latter class is expecially nice because an algebraic poset has a *canonical* (in fact minimal) basis consisting of its finite elements. We can also bring to bear the representation theory discussed in Chapter 2 and get a pleasingly simple class of spaces.

Regretably, however, this class is a bit *too* simple because there is one big problem: the poset of functions between algebraic cpo's may not be algebraic! In light of Theorem 47 we must restrict ourselves down to some subcategory of the profinite domains in order to get algebraic function spaces. But wait, do we always want the *total* function space? If we want closure under the *partial* function space then we demonstrate in this section that a different category of algebraic cpo's is suggested. These are the *pre-domains*. They are very similar to the profinite domains in having pleasing categorical and representational properties. We study them in a manner analogous to our study of the profinites through the equivalent category of Plotkin orders. First we show how to represent continuous partial functions over algebraic cpo's through the use of the following generalization of approximable relations.

Definition: A *partial approximable relation* $\phi : A \rightharpoonup B$ is a subset of $A \times B$ which satisfies the following axioms for any $X, X' \in A$ and $Y, Y' \in B$:

1. if $X \ \phi \ Y$ and $X \ \phi \ Y'$ then there is a $Z \in B$ such that $X \ \phi \ Z$ and $Z \vdash_B Y, Y'$;
2. if $X \vdash_A X' \ \phi \ Y' \vdash_B Y$ then $X \ \phi \ Y$. □

Composition of partial approximable relations is defined in exactly the same way as for (total) approximable relations. With the identity relation defined as before, the pre-orders and partial approximable relations define a category $\mathbf{PO}_\partial$ which has $\mathbf{PO}$ as a sub-category.

Definition: If A and B are pre-orders and $\phi : A \rightharpoonup B$ is a partial approximable relation then define $|\phi| : |A| \rightharpoonup |B|$ by

$$|\phi|(x) \simeq \begin{cases} \{Y \mid X \ f \ Y \text{ for some } X \in x\} & \text{if this set is non-empty;} \\ \text{undefined} & \text{otherwise.} \end{cases}$$ □

Lemma 90 *For every pair A, B of pre-orders, the poset $\mathbf{PO}_\partial(A, B)$ of partial approximable relations between A and B is isomorphic to $[|A| \rightharpoonup |B|]$. Moreover, for a partial approximable relation ϕ, $|\phi|$ is total if and only if ϕ is approximable.*

Proof. The proof that the correspondence $\phi \mapsto |\phi|$ defines an isomorphism is routine. If ϕ is approximable, then by definition $\{Y \mid X \ f \ Y \text{ for some } X \in x\}$ is non-empty for every ideal x. Thus $|\phi|$ is total. On the other hand, if $|\phi|$ is total and $X \in A$ then $|\phi|(\downarrow X)$ is defined so there is a $Y \in |\phi|(\downarrow X)$. Hence $X \ \phi \ Y$ and ϕ is therefore approximable. □

Definition: For a pre-order A, $A \lhd_\perp B$ if and only if $A_\perp \lhd B_\perp$. A *partial Plotkin order* is a pre-order A such that for every finite $u \subseteq A$, there is a finite $B \lhd_\perp A$ with $u \subseteq B$. □

Proposition 91 *A pre-order A is a partial Plotkin order if and only if $A_\perp$ is Plotkin order.*

Proof. To prove necessity ($\Rightarrow$), suppose A is a partial Plotkin order and $u \subseteq A_\perp$ is finite. Then there is a finite $B \lhd_\perp A$ with $u - \{\perp\} \subseteq B$. Hence $u \subseteq B_\perp \lhd A_\perp$. To get sufficiency

($\Leftarrow$), suppose $A_\bot$ is a Plotkin order and $u \subseteq A$ is finite. Then $u \subseteq B \lhd A_\bot$ for some finite B so $u \subseteq (B - \{\bot\}) \lhd_\bot A$. Hence A is a partial Plotkin order. □

Remark: For pre-orders A and B, $A \lhd_\bot B$ if and only if for every $X \in B$, the set $A \cap \downarrow X$ is either empty or directed.

Definition: Let A and B be pre-orders. We define the *partial exponential pre-order*

$$\langle B^{[A]}, \vdash_{B^{[A]}} \rangle$$

as follows:

- $p \in B^{[A]}$ if and only if p is a finite non-empty subset of $A \times B$ such that for every $Z \in A$, the set
$$\{(X, Y) \in p \mid Z \vdash_A X\}$$
is empty or has a maximum with respect to the ordering on $A \times B$.
- $p \vdash_{B^A} q$ if and only if for every $(X, Y) \in q$ there is a pair $(X', Y') \in p$ such that $X \vdash_A X'$ and $Y' \vdash_B Y$. □

The intuition behind the partial exponential is that each $p \in B^{[A]}$ is a finite piece of a partial approximable relation. Note that if $p \in B^{[A]}$ then we have $\{X \mid (X, Y) \in p\} \lhd_\bot A$.

Lemma 92 *If $\phi : A \rightharpoonup B$ is approximable and $M \lhd_\bot A$, $N \lhd_\bot B$ are finite then $\phi \cap (M \times N)$ is an element of $B^{[A]}$.*

Proof. Let $X \in A$ and suppose the set $\{(X', Y') \in p \mid X \vdash X'\}$ is non-empty. Since $M \lhd_\bot A$ there is an $X_0 \in M$ such that $X \vdash_A X_0 \vdash_A M \cap \downarrow X$. If $v = \{Y \in N \mid X_0 \; \phi \; Y\}$ then because ϕ is partial approximable, there is a $Y \in B$ such that $Y \vdash_B v$ and $X_0 \; \phi \; Y$. Since $N \lhd_\bot B$ there is a $Y_0 \in N$ such that $Y \vdash_B Y_0 \vdash_B N \cap \downarrow Y$ Since ϕ is partial approximable we know also that $X_0 \; \phi \; Y_0$. The conditions of 1 in the definition are therefore satisfied. □

Proposition 93 *Let A and B be pre-orders. Then*

1. *If $M \lhd_\bot A$ and $N \lhd_\bot B$ are finite then $N^{[M]} \lhd_\bot B^{[A]}$.*
2. *If A and B are partial Plotkin orders, then $B^{[A]}$ is a partial Plotkin order.*

Proof. 1. Let $p \in B^{[A]}$ and set $q = \{(X, Y) \in M \times N \mid X \; \phi_p \; Y\}$ where

$$\phi_p = \{(X', Y') \in A \times B \mid X' \vdash_A X \text{ and } Y \vdash_B Y' \text{ for some } (X, Y) \in p\}.$$

The relation ϕ_p is partial approximable so $q \in B^{[A]}$ by Lemma 92. It follows directly from the definition of q that $p \vdash_{B^{[A]}} q$. If $p \vdash_{B^{[A]}} r$ and $r \in N^{[M]}$ then $r \subseteq q$ so $q \vdash_{B^{[A]}} r$. Hence $N^{[M]} \lhd_\bot B^{[A]}$.

2. Suppose u is a finite subset of $B^{[A]}$. Since A and B are partial Plotkin orders, there are finite subsets $M \lhd_\bot A$ and $N \lhd_\bot B$ such that

$$\{X \mid (X, Y) \in u \text{ for some } Y \in B\} \subseteq M, \text{ and}$$
$$\{Y \mid (X, Y) \in u \text{ for some } X \in A\} \subseteq N.$$

By 1, $N^{[M]} \lhd_\bot B^{[A]}$. Since $u \subseteq N^{[M]}$ and $N^{[M]}$ is finite the result follows. □

Proposition 94 *Let A and B be pre-orders. Then*

1. *If $M \lhd_\perp A$ and $N \lhd_\perp B$ then $M \times N \lhd_\perp A \times B$.*

2. *If A and B are partial Plotkin orders then $A \times B$ is a partial Plotkin order.*

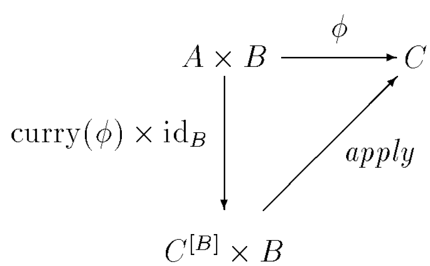
□

For partial Plotkin orders, define the relations apply and curry as they were defined for Plotkin orders. Proofs of the following theorem and its corollary are essentially the same as the proofs of Theorem 15 and Corollary 16 respectively. Note, however, that even for a partial approximable ϕ, curry(ϕ) is total.

Theorem 95 *For any three objects A, B, C in $\mathbf{PLT}_\partial$ the relation*

$$\mathrm{apply} : C^{[B]} \times B \rightharpoonup C$$

is partial approximable and for every $\phi : A \times B \rightharpoonup C$, $\mathrm{curry}(\phi) : A \to C^B$ is the unique approximable relation such that the following diagram commutes



□

Corollary 96 *If A and B are Plotkin orders, then $|B^{[A]}| \cong [|A| \rightharpoonup |B|]$.* □

This almost proves that $\mathbf{PLT}_\partial$ is a cartesian closed category. It fails to prove this, however, because in $\mathbf{PLT}_\partial$, $\times$ *is not a cartesian product!* The arrows fst, snd and the operation $\langle \cdot, \cdot \rangle$ defined in Chapter 2 do not satisfy the necessary commutative diagram condition. To see that no new choices of these arrows will help, we note that in any category, the cartesian product is unique up to isomorphism and show that $\mathbf{PO}_\partial$ has a categorical product which is not isomorphic to $\times$. To this end, let $A \times\times B = A + (A \times B) + B$. Define the arrow pfst : $A \times\times B \rightharpoonup A$ by

- if $X \in A$ and $Y \in A$ then X pfst Y if and only if $X \vdash_A Y$;

- if $(X, Y) \in A \times B$ and $Z \in A$ then (X, Y) pfst Z if and only if $X \vdash_A Z$.

Define psnd : A $\times\times$ B $\rightharpoonup$ B by

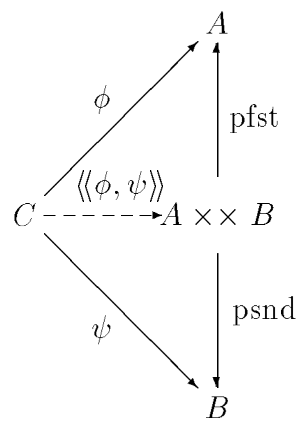
- if $(X, Y) \in A \times B$ and $Z \in B$ then (X, Y) psnd Z if and only if $X \vdash_B Z$;

- if $X \in B$ and $Y \in B$ then X psnd Y if and only if $X \vdash_B Y$.

If $\phi : C \rightharpoonup A$ and $\psi : C \rightharpoonup B$ are partial approximable relations, define a relation $\langle\!\langle \phi, \psi \rangle\!\rangle \subseteq C \times (A \times\times B)$ by

- $X \langle\!\langle \phi, \psi \rangle\!\rangle Y$ for $Y \in A$ if and only if $X \vdash_A Y$ and there is no Z such that $X \; \psi \; Z$;
- $X \langle\!\langle \phi, \psi \rangle\!\rangle (Y, Z)$ for $(X, Z) \in A \times B$ if and only if $X \; \phi \; Y$ and $X \; \psi \; Z$;
- $X \langle\!\langle \phi, \psi \rangle\!\rangle Y$ for $Y \in B$ if and only if $X \vdash_B Y$ and there is no Z such that $X \; \phi \; Z$.

One can show that for any such ϕ and ψ, the relation $\langle\!\langle \phi, \psi \rangle\!\rangle$ is partial approximable and is the unique partial approximable relation which completes the following diagram



Hence $A \times\times B$ is the categorical product of A and B in $\mathbf{PO}_\partial$.
Definition: If $\rho : E \rightharpoonup D$ and $\sigma : D \rightharpoonup E$ are continuous partial functions such that $\rho \circ \sigma = \mathrm{id}_D$ and $\sigma \circ \rho \sqsubseteq \mathrm{id}_E$ then we say that ρ is a partial projection, σ is a partial projection, σ is a partial embedding and we write $\langle \rho, \sigma \rangle : E \overset{\mathrm{pe}}{\rightharpoonup} D$. If $\mathbf{C}$ is a category of cpo's then we denote by $\mathbf{C}^P_\partial$ the category having the same objects as $\mathbf{C}$ but with partial projections as arrows. A dual definition applies to $\mathbf{C}^E_\partial$. □
Definition: A *pre-domain* D is a poset such that $D_\perp$ is profinite. Let **PreDom** be the category whose objects are pre-domains and whose arrows are continuous functions. □

Theorem 97 $\mathbf{PreDom}^P_\partial$ *has limits for inverse systems.*

Proof. The functors $(\cdot)_\perp$ and $(\cdot)_\partial$ define an isomorphism between $\mathbf{PreDom}^P_\partial$ and $\mathbf{BotP}^P$. The theorem therefore follows from Corollary 40. □

Theorem 98 *For a poset D, the following are equivalent*

1. *D is a pre-domain.*
2. *D is isomorphic to the limit in $\mathbf{CPO}^P_\partial$ of an inverse system of finite posets.*
3. *D is isomorphic to the ideal completion of a partial Plotkin order.*

	$\mathbf{PLT}_\partial$	**PreDom**	$\mathbf{BotP}_\perp$
Function space	$B^{[A]}$	**PreDom**(D,E)	$\mathbf{BotP}_\perp(D,E)$
Adjoint to function space	$A \times B$	$D \times E$	$D \otimes E$
Categorical product	$A \times\times B$	$D \times\times E$	$D \times E$
Categorical sum	$A + B$	$D + E$	$D \oplus E$

Table 7.1: Functors on some equivalent categories.

Proof. If D is a pre-domain then $D_\perp$ is profinite so $D_\perp \cong \varprojlim\langle A_i, a_{ij}\rangle$ where $\langle A_i, a_{ij}\rangle$ is an inverse system of finite posets in $\mathbf{CPO}^P$. But each of these A_i has a bottom since $D_\perp$ does, and the maps a_{ij} are strict. Hence $\langle A_i, a_{ij}\rangle$ is an inverse system in $\mathbf{CPO}^P_\perp$ and we have $D \cong D_{\perp\partial} \cong \varprojlim\langle (A_i)_\partial, (a_{ij})_\partial\rangle$. On the other hand, if $\langle A_i, \alpha_{ij}\rangle$ is an inverse system of finite posets in $\mathbf{CPO}^P_\partial$ then $(\varprojlim\langle A_i, \alpha_{ij}\rangle)_\perp \cong \varprojlim\langle (A_i)_\perp, (\alpha_{ij})_\perp\rangle$ which is profinite since each $(A_i)_\perp$ is finite and each $(\alpha_{ij})_\perp$ is a (total) projection. Hence $\varprojlim\langle A_i, \alpha_{ij}\rangle$ is a pre-domain. Thus we have shown that (1) $\Leftrightarrow$ (2).

If D is a pre-domain then $D_\perp$ is profinite so by Theorem 37, $\mathbf{B}[D_\perp]$ is a Plotkin order. But $\mathbf{B}[D_\perp]$=$\mathbf{B}[D]_\perp$ so by definition $\mathbf{B}[D]$ is a partial Plotkin order. Now $D_\perp$ is algebraic so by Lemma 83, D is algebraic and $D \cong |\mathbf{B}[D]|$. Suppose on the other hand that A is a partial Plotkin order. Then $A_\perp$ is a Plotkin order so by Theorem 37, $|A_\perp| \cong |A|_\perp$ is profinite. Hence $|A|$ is a pre-domain. We have shown that (1) $\Leftrightarrow$ (3). □

Corollary 99 *The categories* $\mathbf{PLT}_\partial$ *and* **PreDom** *are equivalent.* □

Now, the category of predomains and the category $\mathbf{BotP}_\perp$ of profinite domains with bottoms and strict functions are isomorphic via the correspondence defined by $(\cdot)_\perp$ and $(\cdot)_\partial$. However, when working with the bottom element in $\mathbf{BotP}_\perp$, functors such as the product cause the bottom to get hopelessly intermingled with the *bona fide* elements. This fact motivates the introduction of the *smash product* $D \otimes E$ which is defined to be the product of D and E with all pairs having bottom in their first or second coordinate identified. Similarly the *coalesced sum* $D \oplus E$ is defined by taking the sum of D and E and identifying their respective bottoms. Table 7.1 relates some of the functors on the categories $\mathbf{PLT}_\partial$, **PreDom** and $\mathbf{BotP}_\perp$. It seems to the author that the functors $\otimes$ and $\oplus$ on $\mathbf{BotP}_\perp$ are less elegant and harder to work with than the corresponding functors $\times$ and $+$ on the other two categories. This is because $\otimes$ and $\oplus$ involve the use of equivalence classes made necessary by the presence of the bothersome bottom element. Whether this is made up for by whatever advantages one preceives strict total functions to have over partial ones is perhaps a matter of application or personal inclination.

Proposition 100 *Open subsets of pre-domains are pre-domains.*

Proof. If $D_\perp$ is profinite and $O \subseteq D$ is open then $O_\perp$ is algebraic. Hence $\mathbf{B}[O_\perp] = O \cap \mathbf{B}[D] \lhd \mathbf{B}[D_\perp]$ so $\mathbf{B}[O_\perp]$ is a Plotkin order. Hence $O_\perp$ is profinite and O is therefore a pre-domain. □

This property sets the pre-domains apart from other categories of algebraic cpo's such as **P** or **BCALG** because these categories are not closed under open subsets (although compact open subsets of profinite domains are profinite and 1-Lindelöf open subsets of objects in **BCALG** are objects of **BCALG**). One can show that when inclusion of open sets on **PreDom** is taken as a *notion of partial* in the sense of Rosolini [1984] the resulting category of partial maps is a category equivalent to $\mathbf{PreDom}_\partial$. This shows that $\mathbf{PreDom}_\partial$ is *dominical*. Moreover, Theorem 95 shows that $\mathbf{PreDom}_\partial$ is a partial cartesian closed category in the sense of Longo and Moggi [1984].

The proof of the following theorem is essentially identical to the proof of 47:

Theorem 101 *If D is an ω-algebraic cpo and $\mathbf{CPO}_\partial(D, D)$ is ω-algebraic then D is a pre-domain.*

Hence the pre-domains arise by analogy with the profinites: as the profinites are to the total functions space, so are the pre-domains to the partial function space. We close the chapter with a proposition which establishes a more direct relationship between pre-domains and profinite domains.

Proposition 102 *The compact pre-domains are exactly the profinite domains.*

Proof. Suppose D is a compact pre-domain. Then $D_\perp$ is profinite and D is a compact open subset of $D_\perp$. But by Corollary 63, compact open subsets of profinite domains are profinite. Thus D is profinite. Since every profinite domain is compact and a pre-domain, the proposition follows. □

References

The following list of references includes *most* of the published references on the theory of domains. In addition, a number of relevant references to expository works in related areas such as first order logic, recursive function theory, category theory, theory of partial orders, and so on are included. Warning: many of the papers and books listed here are not referenced in the text.

ABRAMSKY, S.

1983 *On semantic foundations for applicative multiprogramming* . In: **ICALP 83**, edited by J. Diaz. **Lecture Notes in Computer Science**, vol. 154, Springer-Verlag, 1983, pp. 1–14.

ADÁMEK, J. and KOUBEK, V.

1979 *Least fixed point of a functor.* **Journal of Computer System Sciences**, vol. 19 (1979), pp. 163-178.

ARBIB, M. and MANES, E. G.

1975 **Structures and Functors: the Categorical Imperative**. Academic Press, 1975, 185 pp.

BACK, R. J. R.

1983 *A continuous semantics for unbounded nondeterminism* . **Theoretical Computer Science**, vol. 23 (1983), pp. 187–210.

BARENDREGT, H. P.

1981 **The Lambda Calculus: Its Syntax and Semantics** . North-Holland Publishing Company, 1981, 615 pp.

BARWISE, J.

1977 *An introduction to first order logic.* In: **Handbook of Mathematical Logic**, edited by J. Barwise. North-Holland Publishing Company, 1977, pp. 5–46.

BENSON, D.

1982 *In Scott-Strachey style denotational semantics, parallelism implies nondeterminism.* **Mathematical Systems Theory**, vol. 15 (1982), pp. 267–275.

BIRKHOFF, G.

1940 **Lattice Theory**. A. M. S. Colloquium Publications, 1940, 155 pp.

BRACHO, F.

1983 **Continuously Generated Fixed-Points**. Doctoral Dissertation, Oxford University, 1983, 223 pp.

BROOKES, S.

1984 **Semantics of Programming Languages**. Pre-print, 1984, 200+ pp.

CHANG, C. C. and KEISLER, H. J.

1973 **Model Theory**. North-Holland Publishing Company, 1973, 550 pp.

COPPO, M., DEZANI, M. and LONGO, G.

1983 *Applicative information systems*. In: **Trees in Algebra and Programming**, L'Aquila, March 9–11, 1983, edited by G. Ausiello and M. Protasi. **Lecture Notes in Computer Science**, vol. 159, Springer-Verlag, 1983, pp. 35–64.

ESCRIG, D. F.

1984 *A characterization of Plotkin's order in powerdomains, and some of its properties* . **Theoretical Computer Science**, vol. 31 (1984), pp. 73–82.

GIERZ, G., HOFMANN, K. H., KEIMEL, K., LAWSON, J. D., MISLOVE, M. and SCOTT, D. S.

1981 **A Compendium of Continuous Lattices**. Springer-Verlag, 1981, 371 pp.

GORDON, M.

1979 **The Denotational Description of Programming Languages. An Introduction**. Springer-Verlag, 1979, 271 pp.

HENNESSY, M. and PLOTKIN, G. D.

1979 *Full abstraction for a simple parallel programming language*. In: **MFCS 79**, edited by J. Bečvář. **Lecture Notes in Computer Science**, vol. 74, Springer-Verlag, 1979, pp. 108–120.

HERRLICH, H. and STRECKER, G.

1973 **Category Theory**. Allyn and Bacon Inc., 1973, 400 pp.

HOFMANN, K. H. and MISLOVE, M.

1977 *The lattice of kernal operators*. **Mathematishe Zeitschrift**, vol. 154 (1977), pp. 175–188.

HRBACEK, K.

1984 *Powerdomain construction in the category of algebraic lattices*. Manuscript, City College of New York, 1984, 13 pp.

JOHNSTONE, P. T.

1982 **Stone Spaces**. Cambridge University Press, 1982, 370 pp.

KAMIMURA, T. and TANG, A.

1983 *Kleene chain completeness and fixedpoint properties.* **Theoretical Computer Science**, vol. 23 (1983), pp. 317–331.

1984 *Finitely continuous posets.* Technical Report, no. TR-84-1, University of Kansas, 1984, 26 pp.

KANDA, A.

1979 *Fully effective solutions of recursive domain equations.* In: **MFCS 79**, edited by J. Bečvář. **Lecture Notes in Computer Science**, vol. 74, Springer-Verlag, 1979.

1980 **Effective solutions of recursive domain equations**. Doctoral Dissertation, Warwick University, 1980.

KOYMANS, C. P. J.

1982 *Models of the lambda calculus.* **Information and Control**, vol. 52 (1982), pp. 306–332.

LAMBEK, J. and SCOTT, P. J.

1983 *Aspects of Higher Order Categorical Logic.* Manuscript, 1983, 306 pp.

LAMBEK, J.

1980 *From lambda-calculus to cartesian closed categories.* In: **To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism** , 1980, edited by J. P. Seldin and R. Hindley. Academic Press, 1980, pp. 375–402.

LEHMANN, D. and SMYTH, M. B.

1981 *Algebraic specification of data types: a synthetic approach.* **Mathematical Systems Theory**, vol. 14 (1981), pp. 97–139.

LONGO, G. and MOGGI, E.

1984 *Cartesian closed categories of enumerations for effective type structures (parts I and II).* In: **Semantics of Data Types**, Sophia-Antipolis, June, edited by G. Kahn and G. D. Plotkin. **Lecture Notes in Computer Science**, vol. 173, Springer-Verlag, 1984, pp. 235–256.

MACLANE, S.

1971 **Categories for the Working Mathematician** . Springer-Verlag, 1971, 262 pp.

MACQUEEN, D., PLOTKIN, G. D. and SETHI, R.

1984 *An ideal model for recursive polymorhpic types.* In: **Eleventh Symposium on Principles of Programming Languages**, Salt Lake City, Utah, January 15–18, 1984, edited by K. Kennedy. Association for Computing Machinery, 1984, pp. 165–174.

Main, M. and Benson, D.

1983 *Denotational semantics for "natural" language question-answering programs.* **American Journal of Computational Linguistics**, vol. 9 (1983), pp. 11–21.

Markowsky, G.

1976 *Chain-complete posets and directed sets with applications.* **Algebra Universalis**, vol. 6 (1976), pp. 53–68.

1977 *Categories of chain-complete posets.* **Theoretical Computer Science**, vol. 4 (1977), pp. 125–135.

McCarty, D.

1984 **Realizability and Recursive Mathematics**. Doctoral Dissertation, Merton College, Oxford, 1984, 281 pp.

Also appears as Carnegie-Mellon Department of Computer Science Technical Report number CMU-CS-84-131.

McCracken, N. J.

1982 *A finitary retract model for the polymorphic lambda-calculus.* Manuscript, 1982, 52 pp.

Meyer, A. R.

1982 *What is a model of the lambda calculus.* **Information and Control**, vol. 52 (1982), pp. 87–122.

1984 *Puzzles in the logic of programming.* In: **Workshop on Formal Software Development: Combining Specification Methods**, Nyborg, Denmark, May, edited by D. Bjorner. Springer-Verlag, 1984.

Niño-Salcedo, Jaime

1981 **On Continuous Posets and their Applications**. Doctoral Dissertation, Tulane University, 1981, 105 pp.

Normann, D.

1980 **Recursion on the Countable Functionals**. Springer-Verlag, 1980, 191 pp.

Paola, R. and Heller, A.

1985 *Dominical categories.* Manuscript, 1985, 150 pp.

Plotkin, G. D.

1972 *A set-theoretical definition of application* . Technical Report, no. MIP-R-95, School of Artificial Intelligence, University of Edinburgh , 1972, 32 pp.

1976 *A powerdomain construction.* **SIAM Journal of Computing**, vol. 5 (1976), pp. 452–487.

1978 *The category of complete partial orders: a tool for making meanings..* In: **Proceedings of the Summer School on Foundations of Artificial Intelligence and Computer Science**, Instituto di Scienze dell'Informazione, Universita di Pisa, 1978, 1978.

1978b *T^ω as a universal domain.* **Journal of Computer System Sciences**, vol. 17 (1978b), pp. 209–236.

ROGERS, H.

1967 **Theory of Recursive Functions and Effective Computability** . McGraw Hill Book Company, 1967, 482 pp.

ROSOLINI, G.

1985 *Domains and dominical categories.* Manuscript, 1985, 14 pp.

SANCHIS, L.

1980 *Reflexive domains.* In: **To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism**, 1980, edited by J. P. Seldin and R. Hindley. Academic Press, 1980, pp. 339–361.

SCOTT, D. S.

1972 *Continuous Lattices.* In: **Toposes, Algebraic Geometry and Logic**, edited by F. W. Lawvere. **Lecture Notes in Mathematics**, vol. 274, Springer-Verlag, 1972, pp. 97-136.

1976 *Data types as lattices.* **SIAM Journal of Computing**, vol. 5 (1976), pp. 522–587.

1977 *Logic and programming languages.* **Communications of the ACM**, vol. 20 (1977), pp. 634–641.

1980a *Lambda calculus: some models, some philosophy.* In: **The Kleene Symposium**, Madison, Wisconsin, 1980, edited by J. Barwise. North-Holland Publishing Company, 1980a, pp. 381–421.

1980b *Relating theories of the λ-calculus.* In: **To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism** , 1980, edited by J. P. Seldin and R. Hindley. Academic Press, 1980b, pp. 403–450.

1981a *Lectures on a mathematical theory of computation* . Technical Report, no. PRG-19, Oxford University Computing Laboratory , 1981a, 148 pp.

1981b *Some ordered sets in computer science.* In: **Ordered Sets**, Banff, Canada, August 28 to September 12, 1981, edited by I. Rival. D. Reidel Publishing Company, 1981b, pp. 677–718.

1982a *Domains for denotational semantics.* In: **ICALP 82**, Aarhus, Denmark, July, 1982, edited by M. Nielsen and E. M. Schmidt. **Lecture Notes in Computer Science**, vol. 140, Springer-Verlag, 1982a, pp. 577–613.

1982b *Notes on cpo's/***SFP***-objects and the like* . Manuscript, unpublished, 1982b, 14 pp.

Scott, D. S. and Strachey, C.

1971 *Toward a mathematical semantics for computer languages.* Technical Report, no. PRG-6, Oxford University Computing Laboratory, Programming Research Group, 1971.

Smyth, M. B.

1977 *Effectively given domains.* **Theoretical Computer Science**, vol. 5 (1977), pp. 257–274.

1978 *Power domains.* **Journal of Computer System Sciences**, vol. 16 (1978), pp. 23–36.

1983a *The largest cartesian closed category of domains* . **Theoretical Computer Science**, vol. 27 (1983a), pp. 109–119.

1983b *Power domains and predicate transformers: a topological view.* In: **ICALP 83**, edited by J. Diaz. **Lecture Notes in Computer Science**, vol. 154, Springer-Verlag, 1983b, pp. 662-676.

Smyth, M. B. and Plotkin, G. D.

1982 *The category-theoretic solution of recursive domain equations.* **SIAM Journal of Computing**, vol. 11 (1982), pp. 761–783.

Stoy, J. E.

1977 **Denotational Semantics: The Scott-Strachey Approach to Programming Language Theory**. M.I.T. Press, 1977, 414 pp.

Tarski, A.

1955 *A lattice-theoretical fixedpoint theorem and its applications.* **Pacific Journal of Mathematics**, vol. 5 (1955), pp. 285–309.

Tennent, R. D.

1981 **Principles of Programming Languages**. Prentice-Hall, 1981, 271 pp.

Wadsworth, C.

1976 *The relation between computational and denotational properties for Scott's D_∞ models of the λ-calculus* . **SIAM Journal of Computing**, vol. 5 (1976), pp. 488–521.

Wand, M.

1979 *Fixed point constructions in order-enriched categories.* **Theoretical Computer Science**, vol. 8 (1979), pp. 13-30.

Weihrauch, K. and Deil, T.

1980 *Berechenbarkeit auf cpo's.* Technical Report, no. 63, RWTH, Aachen, 1980, 101 pp.

WEIHRAUCH, K. and SCHÄFER, G.

1983 *Admissable representations of effective cpo's* . **Theoretical Computer Science**, vol. 26 (1983), pp. 131-147.

WINSKEL, G.

1980 **Events in Computation**. Doctoral Dissertation, University of Edinburgh, 1980.

1983 *A note on powerdomains and modality.* In: **FCT 83**, edited by M. Karpinski. **Lecture Notes in Computer Science**, vol. 158, Springer-Verlag, 1983, pp. 505–514.

WINSKEL, G. and LARSEN, K.

1984 *Using information systems to solve recursive domain equations effectively.* In: **Semantics of Data Types**, Sophia-Antipolis, June, edited by G. Kahn and G. D. Plotkin. **Lecture Notes in Computer Science**, vol. 173, Springer-Verlag, 1984, pp. 109–130.

Index