THE LARGEST FIRST-ORDER-AXIOMATIZABLE CARTESIAN CLOSED CATEGORY OF DOMAINS[1]

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Introduction

The inspiration for this paper is a result proved by Michael Smyth which states that Gordon Plotkin's category **SFP** is the *largest cartesian closed category of domains.* Although this category is easily enough motivated from concepts in domain theory and category theory, it is clearly harder to describe and less "elementary" than the most popular categories of domains for denotational semantics. In particular, the category most often used by people who need domain theory is that of *bounded complete algebraic cpo's.* The use of this latter category has been championed by Dana Scott for years ([4], [5], [6]) and its use has become widespread. It is simple to describe, easy to work with, and suffices for most applications.[2]

The purpose of this paper is to state and prove an analog to Smyth's theorem which says that the bounded complete domains form the largest "easy to define" cartesian closed category of domains. Formally, a class $\mathcal{K}$ of domains will be "easy to define" if the posets which form the bases of members of $\mathcal{K}$ are the countable models of a first order theory. This concentration on the bases is reasonable because many domains are best described by explaining what their compact elements are. The domain can then be constructed as the set of ideals of these compact elements ordered by set inclusion. (Indeed, this is a central idea urged in each of Scott's aforementioned papers.) Also, when working with domains, the compact elements are very handy for proving the kinds of facts that one needs to know. So it is important for the compact elements in the domains being used to lie in a familiar, easily-understood class.

The second section of the paper discusses some of the definitions and facts from domain theory and model theory which will be needed. In the third section we define precisely what is meant by a first-order-axiomatizable class of domains and outline the proof of the main result. The proof uses Smyth's Theorem and the Compactness Theorem for first order logic. An alternate proof using ultraproducts is also offered. The final section contains some discussion and a few remarks about possible extensions of the main result.

[1] **Logic in Computer Science,** edited by A. Meyer, IEEE Computer Society Press, June 1986, pp. 42–48.

[2] The well-known exceptions are those applications which use the *convex powerdomain.* In such cases **SFP** is the only satisfactory category known.

Background

Basic Domain Theory. A *poset* is a set D together with a binary relation $\sqsubseteq$ which is reflexive, transitive and anti-symmetric. A set $M \subseteq D$ is *directed* if, for every finite $u \subseteq M$, there is an $x \in M$ such that $y \sqsubseteq x$ for each $y \in u$. A poset D is *complete* (and hence, for short, a *cpo*) if every directed subset M of D has a least upper bound $\bigsqcup M$. We will also assume that a cpo D has a least element. A element $x \in D$ is *compact* if, whenever $x \sqsubseteq \bigsqcup M$ for a directed set M, there is a $y \in M$ such that $x \sqsubseteq y$. Let D^0 denote the set of compact elements of a cpo D. We say that D is *algebraic* if, for every $x \in D$, the set $M = \{y \in D^0 \mid y \sqsubseteq x\}$ is directed and $\bigsqcup M = x$. In other words, in an algebraic cpo every element is the limit of its compact approximations. In this paper we will only be concerned with algebraic cpo's D such that D^0 is countable. So,"algebraic cpo" will always mean the same as "algebraic cpo with a countable basis". To reduce the number of letters needed to write about algebraic cpo's, I will generally just refer to them as "domains".

There a quite a few important operators on cpo's. We will only concern ourselves with two of them—the product and function space. The *product* $D \times E$ is the set of pairs (x, y) with $x \in D$ and $y \in E$ with the coordinatewise ordering. If D and E are cpo's then $D \times E$ will be one also. Moreover, if D and E are *domains,* then $D \times E$ will also be a domain whose basis is $D^0 \times E^0$. A monotone function $f : D \to E$ between cpo's D and E is *continuous* if, for every directed set $M \subseteq D$, $\bigsqcup f(M) = f(\bigsqcup M)$. Let $[D \to E]$ be the set of continuous functions from D to E. We order $[D \to E]$ by setting $f \sqsubseteq g$ if, for every $x \in D$, $f(x) \sqsubseteq g(x)$. It is easy to check that the *function space,* $[D \to E]$, of D and E is itself a cpo.

Let us say that a class $\mathcal{K}$ of domains is *cartesian closed* (and say that $\mathcal{K}$ is a *ccc*) if it contains the one point domain and is closed under products and function spaces.[3] Although the class of all domains *is* closed under products, it is *not* closed under function spaces and is therefore not itself a ccc! It turns out, however, that there is a *largest* ccc of domains. It is defined as follows. Let D be a cpo and let M be the set of continuous functions $p : D \to D$ such that

1. the image of p is finite, and

2. $p(x) = p \circ p(x) \sqsubseteq x$ for each x.

Then D is said to be *strongly algebraic* if M is countable, directed, and $\bigsqcup M$ is the identity function on D. Strongly algebraic domains were introduced in [3] (where they are called "**SFP**-objects"). A proof that the definition given above is equivalent to the original one may be found in [2] where many other properties of strongly algebraic domains are discussed. Of particular interest is the following result of Smyth [7]:

Theorem 1 *If D and $[D \to D]$ are domains, then D is strongly algebraic.* □

[3]This is, of course, a special instance of the categorical notion of cartesian closure.

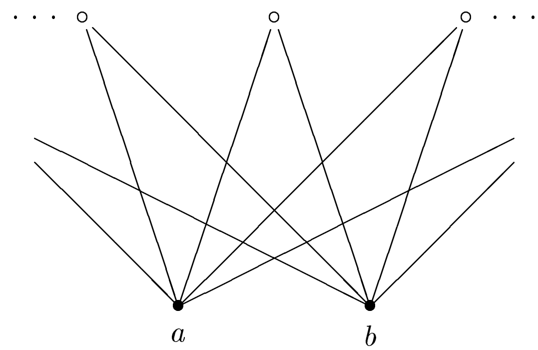

Figure 1: MUB($\{a, b\}$) is not finite

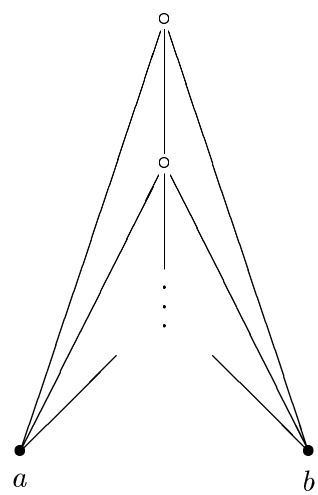

Figure 2: MUB($\{a, b\}$) is not complete

In the next section we will need a technical fact about strongly algebraic domains. Let A be a poset and suppose $u \subseteq A$. An upper bound x for u is said to be *minimal* if, whenever y is an upper bound of u such that $y \sqsubseteq x$, then $x = y$. Let MUB(u) denote the set of minimal upper bounds of u. We say that MUB(u) is *complete* if, whenever x is an upper bound of u, then there is a $y \in$ MUB(u) such that $y \sqsubseteq x$. A proof of the following appears in [3]:

Lemma 2 *If D is strongly algebraic and $u \subseteq D^0$ is finite, then* MUB(u) *is finite and complete.* □

It is easy to show that, in a domain, a minimal upper bound of a finite set of compact elements is itself compact. So it is not necessary to say whether the minimal upper bounds in the conclusion of the lemma are being taken in the domain D or in its basis D^0. The lemma basically rules out configurations such as the ones pictured in figures 1 and 2.

A non-empty domain D is said to be *bounded complete* if every bounded subset of D has a *least* upper bound. One can show that a non-empty domain D is bounded complete iff every bounded pair of elements of D^0 has a least upper bound. The bounded complete domains form a ccc. By Smyth's theorem, it follows that bounded complete domains are strongly algebraic, but this is a fact one could prove easily directly from the definitions. On the other hand, not every strongly algebraic domain is bounded complete. For example, any finite poset (with a least element) is strongly algebraic but there are many such posets that are not bounded complete. In particular, we will need the following fact about the relationship between strongly algebraic domains and bounded complete domains:

Lemma 3 *If D is a strongly algebraic domain that is not bounded complete, then there is a pair of elements $a, b \in D^0$ such that* $\mathrm{MUB}(\{a, b\})$ *has at least two members.*

Proof. Suppose that $\mathrm{MUB}(\{x, y\})$ is empty or a singleton for every $x, y \in D^0$. Suppose $a, b \sqsubseteq c$ for some a, b, c. By Lemma 2, $\mathrm{MUB}(\{a, b\})$ is complete so there is a $d \in \mathrm{MUB}(\{a, b\})$ such that $d \sqsubseteq c$. Since d is the only element of $\mathrm{MUB}(\{a, b\})$ it is the least upper bound of $\mathrm{MUB}(\{a, b\})$. Thus D is bounded complete. □

Basic Model Theory. I will assume that the reader has some familiarity with the kind of first order logic discussed in chapters one and two of [1] such as the notions of language, formula, sentence, model, satisfaction ($\models$), and so on. I will summarize below a few of the results that will be needed in the next section. It will be assumed throughout that the theories T are in a *countable* first order language.

Theorem 4 *(Compactness) A set T of sentences has a countable model iff every finite subset of T has a model.* □

For the definitions of ultrafilter and ultraproduct, see Chapter 4 of [1]. Below we will need the *Fréchet* ultrafilter on ω. This is defined to be the set of subsets $S \subseteq \omega$ such that the complement of S in ω is finite. The primary fact about ultraproducts is the following:

Theorem 5 *(Fundamental Theorem of Ultraproducts) Let U be an ultrafilter over an index set I and suppose $\{A_i \mid i \in I\}$ is an indexed collection of models of a first order theory T. Then the ultraproduct $\prod_U A_i$ is a model of T.* □

The following is a special case of the "downward" Löwenheim-Skolem-Tarski Theorem.

Theorem 6 *Suppose $A \models T$ and $X \subseteq A$ is countable. Then there is a countable submodel $A' \subseteq A$ with $X \subseteq A'$ such that $A' \models T$.* □

Basis-Elementary Classes of Domains

It is now time to say more precisely what I mean by a first-order-axiomatizable class of domains. We will always be working with the first order language of posets expanded, possibly, by countably many constant symbols. The language of posets has one binary relation symbol and the models are posets $\langle D, \sqsubseteq \rangle$ which interpret that symbol as $\sqsubseteq$. A class $\mathcal{K}$ of countable posets is said to be *elementary* if there is a first order theory T in the language of posets such that a countable poset A is in $\mathcal{K}$ iff $A \models T$. If $\mathcal{K}$ is a class of domains, let $\mathcal{K}^0$ be the class of bases D^0 of domains D in $\mathcal{K}$. Let us say that a class of domains $\mathcal{K}$ is *basis-elementary* if $\mathcal{K}^0$ is elementary.

I propose that we consider a class of domains to be *first-order-axiomatizable* just in case it is basis-elementary. Another way that one might define what it means for a class of domains to be first-order-axiomatizable is discussed in the next section. The central theorem of the paper may now be stated as follows:

Theorem 7 *The largest basis-elementary ccc of domains is the class of bounded complete domains.*

Proof using the Compactness Theorem: The reader may check for himself that the bounded complete domains form a basis-elementary class. To show that it is the largest basis-elementary class which is a ccc it suffices to show that a ccc not contained in the bounded complete domains cannot be basis-elementary. Let $\mathcal{K}$ be a ccc of domains which contains a domain that is not bounded complete. Let T be the first order theory of $\mathcal{K}^0$ (*i.e.* T is the set of those sentences ϕ such that $A \models \phi$ for every poset A in $\mathcal{K}^0$). We show that $\mathcal{K}^0$ is not elementary by showing that there is a countable model of T that does not lie in $\mathcal{K}^0$.

Now, $\mathcal{K}$ contains a domain D that is not bounded complete. But D must be strongly algebraic, so by Lemma 3 the basis $A = D^0$ of D contains a pair of elements $a, b \in A$ such that $\mathrm{MUB}(\{a, b\})$ contains more than one member. For each integer $m \geq 2$, we show that there is a model of T in the language of posets expanded by adding two new constant symbols $\mathbf{c}$ and $\mathbf{d}$, that satisfies the first order axiom

$$\phi_m \equiv \exists \mathbf{v}_1 \cdots \exists \mathbf{v}_m . \bigwedge_{i \neq j} \mathbf{v}_i \neq \mathbf{v}_j \wedge \mathbf{v}_i \in \mathrm{MUB}(\{\mathbf{c}, \mathbf{d}\}).$$

Note that A is a model of ϕ_2 if $\mathbf{c}$ and $\mathbf{d}$ are interpreted by a and b. Suppose $m \geq 2$ and $T \cup \{\phi_m\}$ has a model B in which $\mathbf{c}$ and $\mathbf{d}$ are interpreted by c and d. Since $\mathcal{K}$ is a ccc, the product $B \times B$ is in $\mathcal{K}$ and is therefore a model of T. We claim that $B \times B$ is a model of $T \cup \{\phi_{m+1}\}$ when $\mathbf{c}$ and $\mathbf{d}$ are interpreted by (c, c) and (d, d). To see this, note that

$$\mathrm{MUB}(\{(c, c), (d, d)\}) = \mathrm{MUB}(\{c, d\}) \times \mathrm{MUB}(\{c, d\}).$$

There are m^2 elements in the latter set and $m \geq 2$, and hence $m^2 \geq m + 1$ so the claim holds. Now, for each m, $\phi_{m+1} \to \phi_m$ so we may deduce that any finite subset of $T \cup \{\phi_m \mid m \geq 2\}$ has a model. Hence, by the Compactness Theorem, there is a countable model C of $T \cup \{\phi_m \mid m \geq 2\}$.

If C interprets $\mathbf{c}$ and $\mathbf{d}$ by c and d, then $\mathrm{MUB}(\{c,d\})$ is infinite. It follows from Lemma 2 that if E is a domain and $C = E^0$, then E is not strongly algebraic. Since $\mathcal{K}$ is cartesian closed, it follows from Theorem 1 that C does not lie in $\mathcal{K}^0$. Hence $\mathcal{K}$ cannot be basis-elementary. □

Very often, when the Compactness Theorem can be used to show that a class of models is not elementary, it is also possible to prove this fact using the Fundamental Theorem of Ultraproducts. What follows is the sketch of a proof using this technique.

Proof using Ultraproducts. Let $\mathcal{K}$, T, A, a, b be as they were in the proof above and let U be the Fréchet ultrafilter on ω. For each $i \in \omega$, define

$$A^i = \underbrace{A \times \cdots \times A}_{i+1 \text{ copies}}.$$

Elements of the ultraproduct $C = \prod_U A^i$ are equivalence classes of sequences $s : \omega \to \bigcup_i A^i$ such that $s(i) \in A^i$ for each i. In particular, there are sequences

$$a^* = a,\ (a,a),\ (a,a,a),\ \ldots$$
$$b^* = b,\ (b,b),\ (b,b,b),\ \ldots$$

(representing equivalence classes) in C. Now, $\{a,b\}$ has at least two distinct minimal upper bounds p,q in A. Consider the following sequences:

$$\begin{aligned} s_1 &= q,\ (q,p),\ (q,p,p),\ (q,p,p,p),\ \ldots \\ s_2 &= p,\ (p,q),\ (p,q,p),\ (p,q,p,p),\ \ldots \\ s_3 &= p,\ (p,p),\ (p,p,q),\ (p,p,q,p),\ \ldots \\ s_4 &= p,\ (p,p),\ (p,p,p),\ (p,p,p,q),\ \ldots \\ &\vdots \end{aligned}$$

which represent *distinct* elements of C. Each s_i is a minimal upper bound for $\{a^*, b^*\}$ in the ordering on C. By the Fundamental Theorem of Ultraproducts, $C \models T$. Although C may not be countable, by the Löwenheim-Skolem-Tarski Theorem, there is a model $C' \subseteq C$ with $a^*, b^*, s_1, s_2, \ldots \in C'$. such that $C' \models T$. But by Lemma 2, if $C' = E^0$ then E cannot be strongly algebraic. Hence C' does not lie in $\mathcal{K}^0$ and $\mathcal{K}$ cannot be basis-elementary. □

Discussion

There is at least one other reasonable interpretation of "first-order-axiomatizable" class of domains that can be made. Say that a class $\mathcal{K}$ of domains is *intersection-elementary* if there is a theory T of posets such that $\mathcal{K}$ is exactly the class of domains $D \models T$. This notion seems somewhat less natural to me than basis-elementary but I conjecture that the largest intersection-elementary ccc of domains is, in fact, the bounded completes. It should be noted, however, that *no* interesting class of cpo's is actually elementary. For example, any elementary class $\mathcal{K}$ of posets that contains the ordinal $\omega + 1$ also contains the ordinal $\omega + Z + 1$, since these two posets have the same first order theory. But $\omega + Z + 1$ is not a cpo. Thus no elementary class of cpo's can contain $\omega + 1$.

Certainly one consequence of Theorem 7 is that the strongly algebraic domains are not basis-elementary. This does not seem like much of a surprise really since the axioms in the definition of strongly algebraic are clearly higher-order. In [3] there is a condition on the basis of a domain D which is necessary and sufficient for D to be strongly algebraic. Let me explain this condition briefly. If A is a poset we say that a subposet $B \subseteq A$ is *normal* in A if, for every $x \in A$, the set $\{y \in B \mid y \sqsubseteq x\}$ is directed. The following paraphrases the result proved by Plotkin:

Theorem 8 *A domain D is strongly algebraic iff, for every finite subset $u \subseteq D^0$, there is a finite subset $v \subset D^0$ such that v is normal in D^0 and $u \subseteq v$.* $\square$

I think that part of what makes this result appealing is the fact that condition on the right seems less logically complex than the condition given in the earlier definition of a strongly algebraic domain. Instead of quantifying over all functions of a certain kind the condition has its quantifiers ranging over finite subsets of D^0. The theorem shows that although the strongly algebraics are not basis-elementary, their bases can be axiomatized using sentences from *weak* second order logic.

References

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