UNIVERSAL PROFINITE DOMAINS[1]

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Abstract. We introduce a bicartesian closed category of what we call *profinite domains*. Study of these domains is carried out through the use of an equivalent category of pre-orders in a manner similar to the information systems approach advocated by Dana Scott and others. A class of universal profinite domains is defined and used to derive sufficient conditions for the profinite solution of domain equations involving continuous operators. As a special instance of this construction, a universal domain for the category **SFP** is demonstrated. Necessary conditions for the existence of solutions for domain equations over the profinites are also given and used to derive results about solutions of some equations. A new universal bounded complete domain is also demonstrated using an operator which has bounded complete domains as its fixed points.

1 Introduction.

For our purposes a domain equation has the form $X \cong F(X)$ where F is an operator on a class of semantic domains (typically, F is an endofunctor on a category of partially ordered sets). Techniques for solving such equations have been worked out for specific categories (see any of the references by Scott or Plotkin) and in rather general category-theoretic settings as well [28]. Computability has been successfully incorporated into many of these treatments ([29], [8], [9]). All of these approaches use one of three techniques. The most general is the inverse limit construction used by Scott [20] to solve the domain equation $D \cong D \to D$ (where $\cdot \to \cdot$ is the exponential functor). The second uses the Tarski Fixed Point Theorem, which says: if D is a poset with joins for ω-chains and a least element then any function $f : D \to D$ which preserves such joins has a least fixed point. The third—which is introduced in [13]—uses the Banach Fixed Point Theorem, which says: a uniformly contractive function $f : X \to X$ on a non-empty complete metric space X has a unique fixed point. These last

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two approaches employ what is frequently called a "universal" domain to associate with the operator F a join-preserving or contractive map.

In this paper we examine the problems involved in obtaining solutions to equations over the category of profinite domains which will be defined below. This is a rather natural, and in a sense inevitable, category which contains **SFP** (see [17]) as a full subcategory. It has the unusual property of being bicartesian closed, *i.e.* it is cartesian closed and has a coproduct. Such categories have a rich type structure and form models of the typed λ-calculus [11]. Obtaining profinite solutions for domain equations involving the coproduct can be problematic, however. There are categorical impediments to the solution of some equations. For example, the equation $D \cong 1 + (D \rightarrow D)$ (where 1 is the terminal object) has no solution in a *any* non-trivial bicartesian closed category (see [12] and [6]). Moreover, there are equations which have a non-trivial solution in a bicartesian closed category but have no non-trivial solution over the profinites. We provide a condition which, in effect, reduces the problem of solving an equation over the profinite domains to one of getting a finite poset which solves a related equation. This condition is proved sufficient by a variant of the second method described above.

Since no single (projection) universal domain for the profinites exists, we derive an infinite class of domains which are "sufficiently universal" for use in solving equations. Explaining the technique for constructing these domains is the primary goal of the paper. As a secondary theme we show how to extend the neighborhood or information system approach to categories (such as **SFP**) which are larger than the one considered in [22] and [24].

Section two gives some of the basic definitions and explains the equivalence defined by the ideal completion functor. In section three the category of Plotkin orders is introduced and shown to be bicartesian closed. Section four discusses normal substructures and defines the category of profinite domains. Section five contains the primary result of the paper: a technique for constructing universal profinite domains. As a special case the technique provides a universal domain for the category **SFP.** In section six an interesting operator which we call the *join completion* is discussed and used to derive another universal domain. In section seven the universal domains are used to show existence of solutions for a significant class of equations. Section seven also contains discussion of several specific domain equations and their solutions.

2 Pre-orders and Algebraic Dcpo's.

In this section we show how algebraic dcpo's and continuous functions can be represented by pre-orders and *approximable relations.* The idea is to show that something like the notion of an *information system* [24] makes sense for algebraic dcpo's. In the next section we will show how this analogy with information systems can be extended further for a particular subcategory of the pre-orders.

A *pre-order* is a pair $\langle A, \vdash_A \rangle$ where $\vdash_A$ is a binary relation which is reflexive and transitive. It is intended that the "larger" element is the one on the *left* side of the turnstile. We allow $A = \emptyset$. To conserve notation we write $A = \langle A, \vdash_A \rangle$ and when A is clear from context the subscript is dropped. If $X \vdash Y$ and $X \vdash Z$ then we will sometimes write $X \vdash Y, Z$. Indeed, let $f \subseteq A \times B$ be *any* binary relation, then $X\ f\ Y$ means $(X, Y) \in f$. We write $X\ f\ Y, Z$ if $X\ f\ Y$ and $X\ f\ Z$. If $g \subseteq B \times C$ is another binary relation, we write $X\ f\ Y\ g\ Z$ for $X\ f\ Y$ and $Y\ g\ Z$. When the relation f is being considered as an arrow in a category, we write $f : A \to B$ for $f \subseteq A \times B$. The following definition appears in [18] and [25].

Definition: An *approximable relation* $f : A \to B$ is a subset of $A \times B$ which satisfies the following axioms for any $X, X' \in A$ and $Y, Y' \in B$:

1. for every $X \in A$, there is a $Z \in A$ such that $X\ f\ Z$;
2. if $X\ f\ Y$ and $X\ f\ Y'$, then there is a $Z \in B$ such that $X\ f\ Z$ and $Z \vdash_B Y$ and $Z \vdash_B Y'$;
3. if $X \vdash_A X'$ and $X'\ f\ Y'$ and $Y' \vdash_B Y$, then $X\ f\ Y$. □

Let $g : A \to B$ and $f : B \to C$ be approximable relations. We define a binary relation $f \circ g$ on $A \times C$ as follows. For each $X \in A$ and $Z \in C$, $X\ (f \circ g)\ Z$ if and only if there is a $Y \in B$ such that $X\ g\ Y$ and $Y\ f\ Z$. Also, for each pre-order A define $\mathrm{id}_A = \vdash_A$. It is easy to verify that $f \circ g$ and id_A are approximable relations. With this composition and identity relation, the class of pre-orders and approximable relations form a category which we call **PO**. We let **PO**(A, B) be the set of approximable relations on $A \times B$. The approximable relations in **PO**(A,B) are partially ordered by set theoretic inclusion.

For pre-orders A and B define the *product pre-order* to have the coordinatewise ordering:

$$(X, Y) \vdash_{A \times B} (X', Y') \text{ iff } X \vdash_A X' \text{ and } Y \vdash_B Y'.$$

In fact, $\times$ is a categorical product for **PO**. If we take 1 to be a fixed single element pre-order, then, for each pre-order A, there is a unique approximable relation $1_A : A \to 1$. Thus the pre-orders and approximable relations form a cartesian category with terminal object 1. Moreover, the empty poset 0 is initial in this category, *i.e.* for any object A there is a unique arrow $0_A : 0 \to A$. This 0_A is the "empty relation" and it is trivially approximable. For pre-orders A and B, the *coproduct pre-order* $\langle A + B, \vdash_{A+B} \rangle$ is defined by letting $A + B = (A \times \{0\}) \cup (B \times \{1\})$ and defining $(X, n) \vdash_{A+B} (Y, m)$ if and only if either

1. $n = m = 0$ and $X \vdash_A Y$, or

2. $n = m = 1$ and $X \vdash_B Y$.

One can show that $+$ is the *categorical* coproduct in **PO**. This shows that **PO** is *bicartesian*, *i.e.* it has coproduct and initial object as well as product and terminal object.

Let A be a pre-order. A set $S \subseteq A$ is *bounded* if there is an $X \in A$ such that $X \vdash Y$ for every $Y \in S$. Such an X is called a *bound* for S and we write $X \vdash S$. Trivially, any $X \in A$ is a bound for the empty set. A subset $M \subseteq A$ of a pre-order A is *directed* if every finite subset of M has a bound in M. Note, in particular, that every directed set is non-empty. An element $X \in A$ is a *join* of a subset $S \subseteq A$ if, whenever $Y \vdash Z$ for every $Z \in S$, then $X \vdash Y$.

A pre-order $\langle A, \vdash \rangle$ is said to be a *poset* if $\vdash$ is anti-symmetric, *i.e.* if $x \vdash y$ and $y \vdash x$ then $x = y$. When A is a poset we will usually use the symbol $\sqsubseteq$ rather than $\vdash$ for the binary relation. Using the established convention, we write the "larger" element on the *right* side of the $\sqsubseteq$ symbol. If $x \sqsubseteq y$ then it is sometimes convenient to write $y \sqsupseteq x$. If $x \sqsubseteq y$ and $x \neq y$ then we write $x \sqsubset y$; we define $\sqsupset$ by a similar convention. It is frequently desirable to transfer a property of pre-orders to a property of posets and conversely. This is usually possible because pre-orders and posets are closely connected. First of all, every pre-order is isomorphic (in the category with approximable relations as arrows) to a poset. To see this, let $\langle A, \vdash \rangle$ be a pre-order. Define an equivalence relation $\sim$ on A by letting $X \sim Y$ if and only if $X \vdash Y$ and $Y \vdash X$. For each X, let $\tilde{X} = \{Y \in A \mid X \sim Y\}$ and set $\tilde{A} = \{\tilde{X} \mid X \in A\}$. If we define a binary relation $\sqsubseteq$ on A by letting $\tilde{Y} \sqsubseteq \tilde{X}$ if and only if $X \vdash Y$, then it is easy to check that $\langle \tilde{A}, \sqsubseteq \rangle$ is a poset and the approximable relation $f : \tilde{A} \to A$ given by $\tilde{X} \ f \ Y$ if and only if $X \vdash Y$ is an isomorphism. In addition, posets are isomorphic in the category with approximable relations as arrows if and only if they are isomorphic in the more familiar category with monotone maps as arrows.

A poset $\langle D, \sqsubseteq \rangle$ is said to be a *directed complete* (and we call D a "dcpo") if every directed subset $M \subseteq D$ has a join. We will generally use the letters D, E for dcpo's and A, B for pre-orders. If a subset of a poset has a join, then it is unique, and we write $\bigsqcup M$ for the join of M. A monotone function $f : D \to E$ between dcpo's D and E is *continuous* if, for every directed set $M \subseteq D$, $\bigsqcup f(M) = f(\bigsqcup M)$. The dcpo's and continuous functions form a category; we let $\mathrm{id}_D : D \to D$ denote the identity function (context will distintuish this notation from one which uses the approximable relation id_A). Let $D \to E$ be the set of continuous functions from D to E. We order $D \to E$ by setting $f \sqsubseteq g$ if, for every $x \in D$, $f(x) \sqsubseteq g(x)$. It is easy to check that $D \to E$ is itself a dcpo. This definition of dcpo's differs from most other definitions in the literature. We do not require that a dcpo have a least element; indeed, we do not require a dcpo to be non-empty. Much of the usual theory of dcpo's goes through for these "bottomless" cases, but there are some non-trivial differences. For example, a continuous function $f : D \to D$ on a dcpo need not have a fixed point. (However, if $x \sqsubseteq f(x)$ for some $x \in D$, then there is a least $y \sqsupseteq x$ such that $f(y) = y$.)

Let D be a dcpo. An element $x \in D$ is *finite* (or compact) if, whenever $x \sqsubseteq \bigsqcup M$ for a directed set M, there is a $y \in M$ such that $x \sqsubseteq y$. Let $\mathbf{B}_D$ denote the set of finite elements of a dcpo D. We say that D is *algebraic* if, for every $x \in D$, the set $M = \{x_0 \in \mathbf{B}_D \mid x_0 \sqsubseteq x\}$ is directed and $\bigsqcup M = x$. In other words, in an algebraic dcpo every element is the limit of its finite approximations. Let **ALG** be the category of algebraic dcpo's and continuous functions. We now establish an equivalence between **ALG** and **PO**. Suppose $\langle A, \vdash \rangle$ is a pre-order. An *ideal* over A is a directed subset $x \subseteq A$ such that, if $X \vdash Y$ and $X \in x$, then $Y \in x$. The *ideal completion* of A is the partial ordering $\langle |A|, \subseteq \rangle$ of the ideals of A by set-theoretic inclusion. If $X \in A$ then the *principal ideal generated by* X is the set $\downarrow X = \{Y \in A \mid X \vdash Y\}$. Note that $\{\downarrow X \mid X \in A\} \cong A$. We also have the following:

Theorem 1 *If A is a pre-order, then $|A|$ is an algebraic dcpo with $\mathbf{B}_{|A|} = \{\downarrow X \mid X \in A\}$. Moreover, every algebraic dcpo D is representable in this way because $D \cong |\mathbf{B}_D|$.* □

Intuitively, the passage $A \mapsto |A|$ expands A by adding limits for ascending chains. To see this in a simple example, consider the poset ω. The ideal completion adds a limit point and yields $|\omega| = \omega + 1$ as a result. The ideal completion of a countable poset will not always be countable, however. For example, let ${}^{<\omega}2$ be the set of functions $f : n \to 2$ where $n < \omega$. If $f : n \to 2$ and $g : m \to 2$, then say $f \sqsubseteq g$ if and only if $n < m$ and $f(k) = g(k)$ for each $k < n$. The ideal completion $|{}^{<\omega}2|$ of this poset is isomorphic to the union ${}^{<\omega}2 \cup {}^{\omega}2$ where ${}^{\omega}2$ is the set of functions from ω into 2,

- ${}^{<\omega}2$ retains the ordering just mentioned and
- if $f : n \to 2$ and $g : \omega \to 2$ then $f \sqsubseteq g$ if and only if $f(k) = g(k)$ for each $k < n$.

The infinite elements of $|{}^{<\omega}2|$ correspond to those in ${}^{\omega}2$ while the finite elements of $|{}^{<\omega}2|$ correspond to those in ${}^{<\omega}2$. If a poset A has no infinite chains then surely no new elements are added by the ideal completion. We make this intuition precise as follows.

Definition: A poset $\langle A, \sqsubseteq \rangle$ is said to have the *ascending chain condition* (acc) if, for every chain $X_0 \sqsubseteq X_1 \sqsubseteq X_2 \sqsubseteq \cdots$ of elements of A, there is an $n \in \omega$ such that, for every $m \geq n$, $X_m = X_n$. A pre-order $\langle A, \vdash \rangle$ is said to have the acc if $\tilde{A}$ does. □

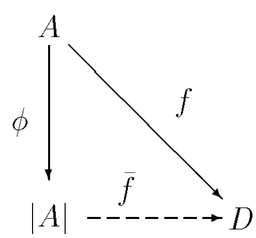
Proposition 2 *If $\langle A, \vdash \rangle$ has the acc then $A \cong |A|$.*

Proof. We show below that $|A| \cong |B|$ if $A \cong B$. Since $A \cong \tilde{A}$ we can therefore assume that A is a poset. We show that each $x \in |A|$ is principal. Assume $x \in |A|$ is *not* principal. Then for each $X \in x$ there is an $X' \in x$ such that $X \sqsubset X'$. But this means there is a chain $X_0 \sqsubset X_1 \sqsubset \cdots$ of elements of x. This contradicts the assumption that A has the acc. Hence $|A| = \{\downarrow X \mid X \in A\} \cong A$. □

A rather obvious corollary of the Proposition is that all finite posets are algebraic dcpo's. Now, if D is a poset with the acc and $M \subseteq D$ is directed, then $\bigsqcup M = x$ for some $x \in M$. Hence, if $f : D \to E$ is monotone then $f(\bigsqcup M) = f(x) = \bigsqcup f(M)$. We conclude that, when D has the acc, $D \to E$ is just the set of monotone functions from D into E.

There is a sense in which $|A|$ is freely generated by A. Formally, we have the following:

Theorem 3 *Let A be a pre-order and suppose $\phi : A \to |A|$ by $\phi : X \mapsto \downarrow X$. Then for every dcpo D and monotone function f, there is a unique continuous function $\bar{f}$ such that the following diagram commutes:*



Moreover, the correspondence $f \mapsto \bar{f}$ is monotone.

Proof. Define $\bar{f}$ by $\bar{f}(x) = \bigsqcup\{f(X) \mid X \in x\}$. □

Definition: If A and B are pre-orders and $f : A \to B$ is an approximable relation, then define a function $|f| : |A| \to |B|$ by $|f|(x) = \{Y \mid X \; f \; Y \text{ for some } X \in x\}$. □

Note that the conditions set down in the definition of an approximable relation insure that the set on the right actually *is* in $|B|$.

Theorem 4 *Let A and B be pre-orders. If $f : A \to B$ is approximable, then $|f| : |A| \to |B|$ is continuous. Moreover, the correspondence $f \mapsto |f|$ is an isomorphism between the posets* $\mathbf{PO}(A, B)$ *and $|A| \to |B|$.*

Proof. To see that $|f|$ is continuous, suppose $M \subseteq |A|$ is directed. Then

$$\begin{aligned}
\bigcup |f|(M) &= \bigcup\{|f|(x) \mid x \in M\} \\
&= \bigcup\{\{Y \mid X \; f \; Y \text{ for some } X \in x\} \mid x \in M\} \\
&= \{Y \mid X \; f \; Y \text{ for some } X \in \bigcup M\} \\
&= |f|(\bigcup M).
\end{aligned}$$

Now, suppose $f : |A| \to |B|$ is continuous. Define a relation $\langle f \rangle \subseteq A \times B$ by letting $X \ \langle f \rangle \ Y$ if and only if $Y \in f(\downarrow X)$. For any $x \in |A|$ we have

$$\begin{aligned} |\langle f \rangle|(x) &= \{Y \mid X \ \langle f \rangle \ Y \text{ for some } X \in x\} \\ &= \{Y \mid Y \in f(\downarrow X) \text{ for some } X \in x\} \\ &= \bigcup \{f(\downarrow X) \mid X \in x\} \\ &= f(x) \end{aligned}$$

since f is continuous. On the other hand, if $f \subseteq A \times B$ is approximable, then $X \ \langle |f| \rangle \ Y$ if and only if $Y \in |f|(\downarrow X)$ if and only if $X \ f \ Y$. Hence $\langle |f| \rangle = f$. Now, if $f \subseteq g$ for approximable relations f and g, then

$$\begin{aligned} |f|(x) &= \{Y \mid X \ f \ Y \text{ for some } X \in x\} \\ &\subseteq \{Y \mid X \ g \ Y \text{ for some } X \in x\} \, . \\ &= |g|(x) \end{aligned}$$

On the other hand, suppose $f, g : |B| \to |A|$ are continuous. If $f \sqsubseteq g$ and $X \ \langle f \rangle \ Y$ then $Y \in f(\downarrow X) \subseteq g(\downarrow X)$ so $X \ \langle g \rangle \ Y$. Hence $\langle f \rangle \subseteq \langle g \rangle$. We conclude that $\mathbf{PO}(A, B) \cong |A| \to |B|$. □

Suppose that $g : A \to B$ and $f : B \to C$ are approximable relations. Then for any $x \in |A|$, one can show that $(|f| \circ |g|)(x) = |f \circ g|(x)$. Since $|\mathrm{id}_A|(x) = x$ for any pre-order A and $x \in |A|$ we may conclude that the passage $A \mapsto |A|$, $f \mapsto |f|$ is a functor. In category theoretic terminology, Theorem 1 says that this functor is *dense* and Theorem 4 says that it is *full* and *faithful*. We have therefore proved the following:

Theorem 5 *The category of pre-orders and approximable relations is equivalent (in the category theoretic sense) to the category of algebraic dcpo's and continuous functions.* □

This equivalence extends to several interesting subcategories as well. If $\mathcal{K}$ is a class of pre-orders then let $\mathbf{Idl}_{\mathcal{K}}$ be the category which has as objects, algebraic dcpo's D such that $\mathbf{B}_D$ is isomorphic to a pre-order in $\mathcal{K}$, and has as arrows, continuous functions. If $\mathcal{K}$ is the class of upper semi-lattices, then $\mathbf{Idl}_{\mathcal{K}}$ is the category of algebraic lattices. Let us say that a non-empty pre-order A is *coherent* if, whenever a finite $u \subseteq A$ is pair-wise bounded, then it has a join. If $\mathcal{K}$ is the class of coherent pre-orders, then it is possible to show that $\mathbf{Idl}_{\mathcal{K}}$ is the category of coherent algebraic dcpo's. A non-empty pre-order is *bounded complete* if each of its finite bounded subsets has a join. Again, if $\mathcal{K}$ is the class of bounded complete pre-orders, then it is possible to show that $\mathbf{Idl}_{\mathcal{K}}$ is the category of bounded complete dcpo's. Each of these three categories is cartesian closed, but none of them has a categorical coproduct. Note also that there is an equivalence between the category of countable pre-orders and the category of countably based algebraic dcpo's.

3 Plotkin Orders.

In this section we introduce the category of *Plotkin orders* which will be our primary technical tool for studying the profinite domains. Plotkin orders are less abstract than profinite domains and in many ways they are easier to work with. For example, Smyth [27] proves many facts about strongly algebraic domains by taking a detailed look at the particular class of Plotkin orders which correspond to such domains. Their use makes some arguments more algebraic and simplifies the definitions of some of the operators (such as the powerdomains) which we discuss later.

Definition: Suppose A is a pre-order and $S \subseteq A$. We say that S is *normal* in A and write $S \lhd A$ if, for every $X \in A$, the set $S \cap \downarrow X$ is directed. □

Note, incidently, that if $S \lhd A$ and $X \in A$, then $X \vdash \emptyset$ (since $X \vdash Y$ for each $Y \in \emptyset$) and $\emptyset \subseteq S$, so there is an $X' \in S$ such that $X \vdash X'$. Let u be a subset of A. A set u' of upper bounds of u is said to be *complete* if, whenever $X \vdash u$, there is an $X' \in u'$ such that $X \vdash X'$. We summarize some more of the properties of the $\lhd$ relation in the following:

Lemma 6 *Let A, B, C be pre-orders.*

1. *Suppose $A \subseteq B$. Then $A \lhd B$ if and only if, for every $u \subseteq A$, there is a set $u' \subseteq A$ of upper bounds for u which is complete for u in B.*
2. *If $A \lhd B \lhd C$ then $A \lhd C$.*
3. *If $A \subseteq B \subseteq C$ and $A \lhd C$ then $A \lhd B$.* □

Definition: A pre-order A is a *Plotkin order* if, for every finite $u \subseteq A$, there is a finite $B \supseteq u$ such that $B \lhd A$. The category of Plotkin orders with approximable relations will be denoted by **PLT**. □

Intuitively, if $S \lhd A$ then S offers a directed approximation to every element of A. Thus one might think of S as itself an approximation to A. A pre-order A is a Plotkin order just in case it can be built up as a directed union of finite approximations. Obviously, any finite pre-order is a Plotkin order. There are a couple of similar conditions on pre-orders which are frequently useful. An upper bound $X \vdash u$ of u is *minimal* if, for each Y, $X \vdash Y \vdash u$ implies $X \sim Y$. If every finite subset of A has a complete set of minimal upper bounds, then we say that A has the *(weak) minimal upper bounds property* (or "property m"). If every finite subset of A has a finite complete set of minimal upper bounds, then we say that A has the *strong* minimal upper bounds property (or "property M"). Any pre-order which has

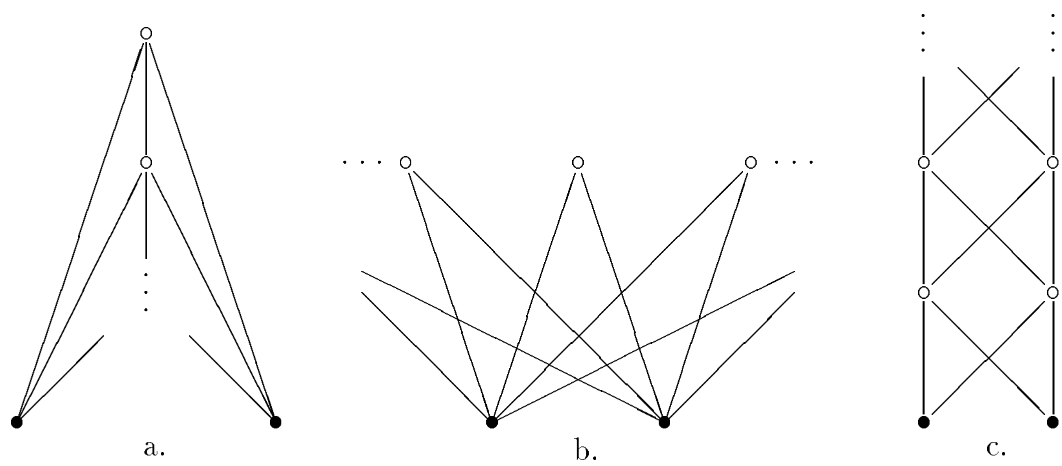


Figure 1: Posets that are not Plotkin orders.

property M and the acc is a Plotkin order. A proof of this uses König's lemma and can be found in [27]. On the other hand,

Proposition 7 *Any Plotkin order has property M.*

Proof. Let A be a poset and suppose $u \subseteq A$ is finite. If a complete set u' of upper bounds of u is finite, then it contains a complete set of minimal upper bounds. If A is a Plotkin order, then there is a finite $B \lhd A$ with $u \subseteq B$. Hence, by Lemma 6, u has a finite set of minimal upper bounds in A. □

It is not true, however, that *every* pre-order having property M is a Plotkin order. A counter-example is illustrated in Figure 1a. Figures 1b and 1c illustrate two other ways in which a pre-order can fail to be a Plotkin order (by failing to have property M).

It is often easier to work with Plotkin orders which are *posets*. Little is lost by this restriction, since every pre-order is isomorphic (in the category with approximable relations as arrows) to a poset $\tilde{A}$ and one can use the axiom of choice to show the following:

Lemma 8 *A pre-order A is a Plotkin order if and only if $\tilde{A}$ is a Plotkin poset.* □

We might have taken the Plotkin posets as our fundamental notion but this would complicate the definitions of some operators and narrow the scope of discussion unnecessarily. However, we will frequently restrict attention to posets in order to simplify the discussion. Suppose A is a pre-order. For each $u \subseteq A$, let

$$\mathrm{MUB}_A(u) = \{X \in A \mid X \text{ is a minimal upper bound of } u\}.$$

For each $S \subseteq A$, we define subsets $\mathcal{U}_A^n(S) \subseteq A$, $n \in \omega$, as follows:

$$\mathcal{U}_A^0(S) = S,$$
$$\mathcal{U}_A^{n+1}(S) = \{X \mid X \in \mathrm{MUB}_A(u) \text{ for some finite } u \subseteq \mathcal{U}_A^n(S)\},$$
$$\mathcal{U}_A^*(S) = \bigcup_{n\in\omega} \mathcal{U}_A^n(S).$$

As usual, when A is understood from context we drop the subscripts.

Lemma 9 *If A is a poset with property m and $S \subseteq A$, then*

$$\mathcal{U}^*(S) = \bigcap\{B \mid S \subseteq B \lhd A\} \lhd A.$$

Thus, A is a Plotkin poset if and only if A has property m and for every finite $u \subseteq A$, $\mathcal{U}^*(u)$ is finite.

Proof. Suppose $S \subseteq B \lhd A$. Then clearly $S = \mathcal{U}^0(S) \subseteq B$. So suppose $\mathcal{U}^n(S) \subseteq B$ and $X \in \mathrm{MUB}(u)$ for some finite $u \subseteq \mathcal{U}^n(S)$. Since $B \lhd A$, there is a $Y \in B$ such that $X \sqsupseteq Y \sqsupseteq u$. But this means $Y = X$, so $X \in B$. Hence $\mathcal{U}^{n+1}(S) \subseteq B$ and we conclude that $\mathcal{U}^*(S) \subseteq B$. To see that $\mathcal{U}^*(S) \lhd A$, let $u \subseteq \mathcal{U}^*(S)$ be finite. Then $u \subseteq \mathcal{U}^n(S)$ for some n. So, if $X \sqsupseteq u$ then $X \sqsupseteq Y$ for some $Y \in \mathrm{MUB}(u) \subseteq \mathcal{U}^{n+1}(X) \subseteq \mathcal{U}^*(S)$. □

Definition: Let A and B be pre-orders. We define the *exponential pre-order* $\langle B^A, \vdash_{B^A}\rangle$ as follows:

1. $p \in B^A$ if and only if p is a finite non-empty subset of $A \times B$ such that, for every $Z \in A$, the set $\{(X,Y) \in p \mid Z \vdash_A X\}$ has a maximum with respect to the ordering on $A \times B$.
2. $p \vdash_{B^A} q$ if and only if, for every $(X,Y) \in q$, there is a pair $(X',Y') \in p$ such that $X \vdash_A X'$ and $Y' \vdash_B Y$. □

The intuition behind the exponential is that each $p \in B^A$ is a finite piece of an approximable relation. The complexity of condition 1 is due to the fact that p must contain enough information to specify what is happening at the minimal upper bounds of finite subsets of A. This is essential if p is to correspond to a unique continuous function. In terms of our notation: if $p \in B^A$ then $\{X \mid (X,Y) \in p\} \lhd A$. It helps to understand the elements of B^A in terms of the familiar concept of a *step function.* If $p \in B^A$, define $\mathrm{step}_p : \tilde{A} \to \tilde{B}$ by

$$\mathrm{step}_p(\tilde{Z}) = \max\{\tilde{Y} \mid Z \vdash X \text{ and } (X,Y) \in p\}.$$

Then step_p is a monotone function and, for each $p, q \in B^A$, $\mathrm{step}_p \sqsupseteq \mathrm{step}_q$ if and only if $p \vdash_{B^A} q$.

If we "order" the posets with the relation $\triangleleft$ then we come quite close to getting a dcpo. The relation $\triangleleft$ is reflexive (on posets), anti-symmetric and transitive. Moreover, if M is a collection of posets which is directed with respect to $\triangleleft$, then $\bigcup M$ is the join of M. The only reason that the posets with $\triangleleft$ *fail* to be a dcpo is that the posets form a proper class—not a set. When we think about **PO** as ordered by $\triangleleft$ we lose anti-symmetry. But this is a small matter; the following definitions of monotone and continuous operators still seem quite natural.

Definition: Suppose $\mathbf{C} \subseteq \mathbf{PO}$. Let us say that an operator $F : \mathbf{C} \to \mathbf{C}$ is *monotone* if, for every pair of pre-orders $A \triangleleft B$, we have $F(A) \triangleleft F(B)$. A monotone operator is *continuous* if, for every pre-order A and directed set $\mathcal{M}$ of normal substructures of A such that $A = \bigcup \mathcal{M}$, we have $F(A) = \bigcup\{F(B) \mid B \in \mathcal{M}\}$. □

It is possible to link continuity in the sense of the above definition to continuity in the categorical sense by thinking of the pre-orders as a category with the relations $\triangleleft$ as arrows. Then the monotone operators are functors and the continuous operators are functors which preserve filtered colimits. Later we show how to find fixed points for continuous *operators* in a manner analogous to that used for finding fixed points of continuous *functions.* But there is another use of the continuity condition on operators given by the following:

Theorem 10 *If $F : \mathbf{PO} \to \mathbf{PO}$ is a continuous operator and $F(A)$ is a Plotkin order whenever A is finite, then F cuts down to an operator on* **PLT**, *i.e. $F(A)$ is a Plotkin order whenever A is a Plotkin order.*

Proof. Suppose A is a Plotkin order and $u \subseteq F(A)$ is finite. Let $\mathcal{M} = \{B \triangleleft A \mid B \text{ is finite}\}$. Since A is a Plotkin order, this set is directed and $\bigcup \mathcal{M} = A$. Hence, by the continuity of F, $F(A) = \bigcup \mathcal{M}'$ where $\mathcal{M}' = \{F(B) \mid B \in \mathcal{M}\}$. Since u is finite and $\mathcal{M}'$ is directed, there is a $B \triangleleft A$ such that $u \subseteq F(B)$. Now, $F(B)$ is a Plotkin order, so there is a finite subset $C \triangleleft F(B)$ with $u \subseteq C$. But $F(B) \triangleleft F(A)$ since F is monotone, so we must also have $C \triangleleft F(A)$. Hence $F(A)$ is a Plotkin order. □

The definition and theorem can be extended in a straight-forward way to include multiary operators. If $F : \mathbf{PO}^n \to \mathbf{PO}$ then say that F is monotone (continuous) if it is monotone (continuous) in each of its n coordinates. If $F : \mathbf{PO}^n \to \mathbf{PO}^m$ by

$$F(A_1, \ldots, A_n) = (G_1(A_1, \ldots, A_n), \ldots, G_m(A_1, \ldots, A_n))$$

then say that it is monotone (continuous) if G_i is monotone (continuous) for each $i = 1, \ldots, m$. It is easy to check that composition of operators preserves monotonicity and continuity. We have the following:

Corollary 11 *The product and coproduct operators are continuous and send finite pre-orders to finite pre-orders. Hence they cut down to operators on* **PLT**. □

Unfortunately, Theorem 10 is not quite general enough to apply to the exponential operator. So we treat the exponential separately below. The following lemma is technically useful and helps pin down the intuition behind the definition of B^A:

Lemma 12 *If $f : A \to B$ is an approximable relation and $M \lhd A$, $N \lhd B$ are finite, then $f \cap (M \times N)$ is an element of B^A.*

Proof. Let $X \in A$. Since $M \lhd A$ there is an $X_0 \in M$ such that $X \vdash_A X_0 \vdash_A M \cap {\downarrow}X$. If $v = \{Y \in N \mid X_0\ f\ Y\}$ then, because f is approximable, there is a $Y \in B$ such that $Y \vdash_B v$ and $X_0\ f\ Y$. Since $N \lhd B$ there is a $Y_0 \in N$ such that $Y \vdash_B Y_0 \vdash_B N \cap {\downarrow}Y$. But f is approximable so $X_0\ f\ Y_0$. □

Theorem 13 *Let A and B be pre-orders.*

1. *If $M \lhd A$ and $N \lhd B$ are finite, then $N^M \lhd B^A$.*
2. *If A and B are Plotkin orders, then B^A is a Plotkin order.*

Proof. 1. Let $p \in B^A$ and set $q = \{(X, Y) \in M \times N \mid X\ f_p\ Y\}$ where

$$f_p = \{(X', Y') \in A \times B \mid X' \vdash_A X \text{ and } Y \vdash_B Y' \text{ for some } (X, Y) \in p\}.$$

We check the three conditions for approximablility of f_p. First, if $X \in A$ then there is an $(X', Y') \in p$ such that $X \vdash_A X'$. Hence $X\ f_p\ Y'$. For the second condition, suppose $X\ f_p\ Y_0$ and $X\ f_p\ Y_1$. Let $(X_0', Y_0'), (X_1', Y_1') \in p$ be such that $X \vdash_A X_0', X_1'$ and $Y_0' \vdash_B Y_0$ and $Y_1' \vdash_B Y_1$. Since $p \in B^A$, there is a pair $(X', Y') \in p$ such that $X \vdash_A X'$ and $X' \vdash_A X_0', X_1'$ and $Y' \vdash_B Y_0', Y_1'$. Hence $X\ f_p\ Y'$ and $Y' \vdash_B Y_0, Y_1$. To get the third condition note that, if $X \vdash_A X'$ and $X'\ f_p\ Y'$ and $Y' \vdash_B Y$, then $X\ f_p\ Y$ follows immediately from the definition of f_p. Since f_p is approximable, $q \in B^A$ by Lemma 12. It follows directly from the definition of q that $p \vdash_{B^A} q$. If $p \vdash_{B^A} r$ and $r \in N^M$ then $r \subseteq q$ so $q \vdash_{B^A} r$. Hence $N^M \lhd B^A$.

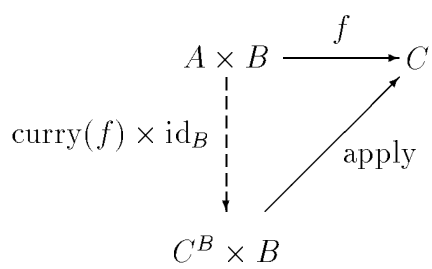
2. Suppose u is a finite subset of B^A. Since A and B are Plotkin orders, there are finite subsets $M \lhd A$ and $N \lhd B$ such that

$$\{X \mid (X, Y) \in u \text{ for some } Y \in B\} \subseteq M, \text{ and}$$
$$\{Y \mid (X, Y) \in u \text{ for some } X \in A\} \subseteq N.$$

By 1, $N^M \lhd B^A$. Since $u \subseteq N^M$ and N^M is finite, the result follows. □

We now arrive at the central fact about the exponential and product on **PLT**.

Definition: A bicartesian category **C** is *closed* if there is a (specified) binary operation B^A such that, for any triple A, B, C of **C**-objects, there is an arrow apply : $C^B \times B \to C$ such that, for every $f : A \times B \to C$, there is a unique arrow curry$(f) : A \to C^B$ which makes the following diagram commute:



Theorem 14 *The category* **PLT** *is bicartesian closed.*

Proof. We have already shown that **PO** is bicartesian. By Corollary 11 the product and coproduct are endofunctors on **PLT**. Since 0 and 1 are finite they are Plotkin orders so **PLT** must be bicartesian. Theorem 13 says that the exponential is defined on **PLT**. To complete the proof we must demonstrate the maps curry and apply. For pre-orders B and C, define apply $\subseteq (C^B \times B) \times C$ by

$$(p, X) \text{ apply } Y \text{ iff } \exists (X', Y') \in p.\ X \vdash_A X' \text{ and } Y' \vdash_B Y.$$

Suppose $X \in A$ and $p \in C^B$. If $f : A \times B \to C$, define curry(f) by

$$X \text{ curry}(f)\ p \text{ iff } \forall (Y, Z) \in p.\ (X, Y)\ f\ Z.$$

One can show that curry and apply are approximable. To see that apply $\circ$ (curry$(f) \times \mathrm{id}_B) =$ f, take $(X, Y) \in A \times B$ and $Z \in C$ such that $(X, Y)\ f\ Z$. Using the fact that C and B are Plotkin orders one can show that there is a $p \in C^B$ with $(Y, Z) \in p \subseteq f$. Thus X curry(f) p and (p, Y) apply Z, so

$$(X, Y) \text{ apply} \circ (\text{curry}(f) \times \mathrm{id}_B)\ Z. \qquad (*)$$

On the other hand, suppose equation $(*)$ holds. Then there is a $p \in C^B$ such that X curry(f) p and (p, Y) apply Z. By the definition of apply, there is a pair $(Y', Z') \in p$ such that $Y \vdash_B Y'$ and $Z' \vdash_C Z$ and $(X, Y')\ f\ Z'$. Now, X curry(f) p implies $(X, Y')\ f\ Z'$. Hence $(X, Y)\ f\ Z$. We leave the proof that curry(f) is unique to the reader. □

Corollary 15 *If A and B are Plotkin orders, then* $|B^A| \cong |A| \to |B|$.

Proof. By Theorem 4, $|A| \to |B| \cong \mathbf{PO}(A,B)$. It is also clear that $\mathbf{PO}(A,B) \cong \mathbf{PO}(1 \times A, B)$ and $\mathbf{PO}(1, B^A) \cong |B^A|$. By Theorem 14, curry : $\mathbf{PO}(1 \times A, B) \to \mathbf{PO}(1, B^A)$ is a bijection with inverse $g \mapsto \mathrm{apply} \circ (g \times \mathrm{id})$. The fact that curry and its inverse are monotone follows immediately from their definitions, so we have the desired isomorphism. □

The assumption in the corollary that A and B are Plotkin orders is important. The result does *not* hold for all pre-orders. Now, let ω-**PLT** be the category of countable Plotkin orders and approximable relations. It is easy to see that $A \times B$, $A + B$ and B^A are all countable whenever A and B are countable. We therefore have the following:

Theorem 16 ω-**PLT** *is bicartesian closed.* □

4 Projections and Profinite Domains.

Categorically speaking, a dcpo is profinite if it is isomorphic to an inverse limit of finite posets in the category of dcpo's with projections as arrows. We explain shortly what a projection is, but we hope to circumvent the use of this categorical definition in favor of notions which are more elementary and intrinsic. Profinite domains with a countable basis (which we will call *ω-profinite domains*) and least element are called *strongly algebraic* domains.[3] With continuous functions as arrows, they form a cartesian closed category called **SFP** which was introduced by Gordon Plotkin [17]. To the reader familiar with these, a countably based profinite domain is a poset which is isomorphic to a Scott compact open subset of a strongly algebraic domain. In other words, a poset D is ω-profinite if and only if there is a strongly algebraic poset E and a finite set $u \subseteq \mathbf{B}_E$ such that $D \cong \{x \in E \mid x \sqsupseteq y \text{ for some } y \in u\}$. Thus, if D is ω-profinite then the lift[4] $D_\perp$ of D is strongly algebraic. However, it is *not* true, in general, that if $D_\perp$ is strongly algebraic then D is ω-profinite.

Let D and E be dcpo's. A *projection-embedding pair* is a pair $\langle p, q\rangle$ of continuous maps $p : E \to D$ and $q : D \to E$ such that $p \circ q = \mathrm{id}_D$ and $q \circ p \sqsubseteq \mathrm{id}_E$. The function p is the projection and q is the embedding. We abbreviate by writing $\langle p, q\rangle : E \xrightarrow{\mathrm{pe}} D$. In this section we look at the relationship between normal substructures of pre-orders and pe-pairs from the point of view of approximable relations. We thereby generalize the theory exposited in [23] to the category of algebraic dcpo's. These results will be used to derive a universal domain technique for the Plotkin orders. Let A and B be pre-orders. Write $A \trianglelefteq B$ if there is an $A' \triangleleft B$ such that $A \cong A'$.

[3] As far as the author knows, this terminology was first used in [27].

[4] The *lift* of D is obtained by attaching a new element $\perp$ to D which is taken to lie below each of the elements of D.

Theorem 17 *Let A and B be pre-orders.*

1. *Suppose $A \triangleleft B$ and $\vdash$ is the order relation on $B \times B$. If $p = (B \times A) \cap \vdash$ and $q = (A \times B) \cap \vdash$ then p, q are approximable relations, $p \circ q = \mathrm{id}_A$ and $q \circ p \subseteq \mathrm{id}_B$. In other words $\langle |p|, |q| \rangle : |B| \xrightarrow{\mathrm{pe}} |A|$.*

2. *Conversely, if $\langle |p|, |q| \rangle : |B| \xrightarrow{\mathrm{pe}} |A|$ for approximable relations p and q, then $A \underset{\sim}{\triangleleft} B$. In particular, $A \cong A' = \{Y \in B \mid Y \ (q \circ p) \ Y\} \triangleleft B$.*

Proof. The proof of 1 is a straight-forward verification. To prove 2, we begin by showing that $A' \triangleleft B$. Suppose $u \subseteq A'$ is finite and $Z \vdash u$. For each $X \in u$, there is an $X' \in A$ such that $X \ p \ X' \ q \ X$. Let $v = \{X' \mid X \in u\}$. Then $Z \ p \ X'$ for each $X' \in v$ so there is a $Y \in A$ such that $Z \ p \ Y \vdash v$. Now, $Y \ p \circ q \ Y$ so there is a $Z' \in B$ such that $Y \ q \ Z' \ p \ Y$. But then $Z' \ p \ Y \ q \ Z'$ so $Z' \in A'$. If $X \in u$ then $Y \vdash X'$ so $Y \ q \ X$. Since $Z' \ p \ Y$ we get $Z' \ q \circ p \ X$ and therefore $Z' \vdash X$. Moreover, $Z \ p \ Y \ q \ Z'$ so $Z \vdash Z'$.

Let $p' = p \cap (A' \times A)$ and $q' = q \cap (A \times A')$. That p' is approximable follows immediately from the approximability of p. If $X \in A$ and $X \ q' \ Y, Y'$ for $Y, Y' \in A'$, then $X \ q \ Z$ for some $Z \in B$ such that $Z \vdash Y, Y'$. Since $A' \triangleleft B$, there is a $Z' \in A'$ such that $Z \vdash Z' \vdash Y, Y'$. Hence $X \ q' \ Z'$. The other conditions are easy to check. Now, suppose $X \in A$. Then $X \ p \circ q \ X$ so $X \ q \ Y \ p \ X$ for some $Y \in B$. But then $Y \in A'$ so $X \ p' \circ q' \ X$. Since $p' \circ q' \subseteq \mathrm{id}_A$, we conclude that $p' \circ q' = \mathrm{id}_A$. Suppose, on the other hand, that $Y \in A'$. Then, by definition, $Y \ q \circ p \ Y$. Since $q' \circ p' \subseteq \mathrm{id}_{A'}$ we must have $q' \circ p' = \mathrm{id}_{A'}$. This proves the desired isomorphism. □

Given a function $g : D \to E$, let $im(g) = \{f(x) \mid x \in D\}$.

Theorem 18 *Suppose A is a pre-order and $f : A \to A$ is an approximable relation such that $f \circ f = f \subseteq \mathrm{id}_A$. Then the following are equivalent:*

1. *$im(|f|)$ is algebraic.*

2. *For each $X, Z \in A$, if $X \ f \ Z$ then $X \vdash Y \ f \ Y \vdash Z$ for some $Y \in A$.*

Proof. (1) $\Rightarrow$ (2). Suppose $X \ f \ Z$. Then $Z \in |f|(\downarrow X)$ and since $im(|f|)$ is algebraic there is a finite $x \in im(|f|)$ such that $Z \in x \subseteq |f|(\downarrow X)$. But x is finite in $|A|$ so $x = \downarrow Y$ for some Y. This Y has the property in the conclusion of (2).

(2) $\Rightarrow$ (1). If $X \ f \ X$ then $\downarrow X = |f|(\downarrow X)$ so $\downarrow X$ is a finite element of $im(|f|)$. If $x \in |A|$ then

$$\begin{aligned} |f|(x) &= \{Z \mid X \ f \ Z \text{ some } X \in x\} \\ &= \{Z \mid X \vdash Y \ f \ Y \vdash Z \text{ some } X \in x \text{ and some } Y\} \\ &= \bigcup \{\downarrow Y \mid Y \in x \text{ and } Y \ f \ Y\}. \end{aligned}$$

To see that this set is directed, suppose $X\ f\ X$ and $Y\ f\ Y$. If $Z\ f\ X, Y$ then $Z\ f\ Z' \vdash X, Y$ for some Z'. Hence $Z \vdash W\ f\ W \vdash Z' \vdash X, Y$ for some W. We conclude that $im(|f|)$ is algebraic. □

Definition: If A is a poset then we denote by $N(A)$ the set of normal substructures of A, ordered by set inclusion. □

Proposition 19 *Let A be a poset. Then $N(A)$ is a dcpo. If A has property m, then $N(A)$ has a least element called the root of A. It is given by the equation* $\mathrm{rt}(A) = \bigcap\{B \mid B \lhd A\}$.

Proof. Suppose $\mathcal{M} \subseteq N(A)$ is directed and $X \in A$. If $u \subseteq \downarrow X \cap (\bigcup \mathcal{M})$ is finite then $u \subseteq B$ for some $B \in \mathcal{M}$. Since $B \lhd A$ there is an $X' \in B$ such that $X \vdash X' \vdash u$. Hence $\bigcup \mathcal{M} \in N(A)$. Obviously, $\bigcup \mathcal{M}$ is the join of $\mathcal{M}$. Now suppose A has property m. Note that if $u \subseteq \mathrm{rt}(A)$ is finite then the complete set u' of minimal upper bounds of u is in B for each $B \lhd A$. Hence $\mathrm{rt}(A) \lhd A$. $\mathrm{rt}(A)$ is evidently the least member of $N(A)$. □

Actually, if A has property m then $N(A)$ is an algebraic lattice. And if A is a Plotkin poset then $N(A)$ is a locally finite algebraic lattice; that is, $\{x_0 \in \mathbf{B}_{N(A)} \mid x_0 \sqsubseteq x\}$ is finite for each $x \in \mathbf{B}_{N(A)}$. Later we will need the following:

Lemma 20 *Let A and B be posets and suppose $i : A \trianglelefteq B$. Then the function $N(i) : N(A) \to N(B)$ given by $N(i)(A') = \{i(X) \mid X \in A'\}$ is continuous.* □

For the puposes of this paper it is easiest to define profinite domains as follows:

Definition: A dcpo D is *profinite* if it is isomorphic to the ideal completion of a Plotkin order A. If A is countable, then D is said to be ω-profinite. □

By Theorem 5, we know that the category of profinite domains and continuous functions is equivalent to **PLT**. This equivalence cuts down to an equivalence between ω-profinite domains and ω-**PLT**. There are several other ways of characterizing the profinite domains; two of these were mentioned at the beginning of the section. The definition above was chosen because it is the best suited for the constructions in the next section. The reader is refered to [4] for a full discussion.

5 Universal Domains.

We now investigate the mathematical problem of the existence of a profinite universal domain. In the literature there are three primary examples of universal domains. The simplest is the so-called *graph model* $\mathbf{P}\omega$ which is the algebraic lattice of subsets of ω, ordered by set

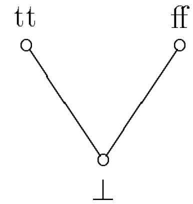


Figure 2: The truth value dcpo.

inclusion. It receives a detailed study in [21] where it is proved that any countably based algebraic lattice is a retract of $\mathbf{P}\omega$.[5] Some domain theorists felt, however, that for applications in denotational semantics of programming languages it would be easier to use a class which did not require the existence of a largest (top) element. Plotkin [19] showed that the poset $\mathbb{T}^\omega$ of functions from ω into the truth value dcpo $\mathbb{T}$ (see figure 2) is universal in the sense that every coherent ω-algebraic dcpo is a retract of $\mathbb{T}^\omega$. Since $\mathbb{T}^\omega$ is itself algebraic and coherent this provided a universal domain for a class of algebraic dcpo's that included the algebraic lattices but also contained certain desirable dcpo's without tops. In [22], [24] and [23], a third universal domain $\mathbb{U}$ is discussed. Although $\mathbb{U}$ is harder to understand than $\mathbf{P}\omega$ or $\mathbb{T}^\omega$ it has the advantage of having every bounded complete ω-algebraic dcpo as a *projection* (not just as a retract). There are instances in which a "retraction universal" domain does not have all of the desired properties so that a "projection universal" domain is needed. For example Mulmuley [14] requires a projection universal domain to prove some of his results on the existence of inclusive predicates (for showing equivalence of semantics). Table 1 lists some of the known results on universal domains. Posets in the left column are assumed to be countable; their ideal completions are countably based.

Elementary proofs of the universality of $\mathbb{U}$ appear in [22] and in [3]. A less elementary proof which uses results from the previous section can be carried out as follows. Let $\mathbb{B}$ be the countable atomless boolean algebra and suppose A is a countable bounded complete poset. Now, A can be embedded into a countable boolean algebra in a way that preserves existing joins in A and such that the join of the image of an unbounded subset of A is the top element. But any countable boolean algebra is isomorphic to a subalgebra of $\mathbb{B}$. Thus $A \trianglelefteq \mathbb{B}^-$ where $\mathbb{B}^-$ is $\mathbb{B}$ minus its top element. We conclude that, if A is countable and bounded complete, then there is a continuous projection $p : |\mathbb{B}^-| \to |A|$. Thus $\mathbb{U} = |\mathbb{B}^-|$ is universal for the bounded complete algebraic dcpo's.

In what follows we use a technique similar to the one for $\mathbb{U}$ to get universal domains for certain classes of ω-profinite domains. If A is a poset with property m, then we remarked in Lemma 19 that $\mathrm{rt}(A)$ is the least element in $N(A)$. Now, if A and B are Plotkin posets

[5] A continuous function $r : E \to D$ is said to be a *retraction* if there is a continuous function $r' : D \to E$ (called a *section*) such that $r \circ r' = \mathrm{id}_D$. If there is a retraction $r : E \to D$ then D is said to be a *retract* of E.

POSETS	IDEAL COMPLETIONS	UNIVERSAL DOMAIN
Upper Semi-lattices	Algebraic Lattices	$\mathcal{P}\omega$
Coherent Pre-orders	Coherent Algebraic Dcpo's	$\mathbb{T}^{\omega}$
Bounded Complete Pre-orders	Bounded Complete Algebraic Dcpo's	$\mathbb{U}$
Plotkin Orders	Profinite Domains	?

Table 1: Universal domains.

and $A \underset{\sim}{\triangleleft} B$, then $\mathrm{rt}(A) \cong \mathrm{rt}(B)$. Hence, by Theorem 17, no profinite domain can be a continuous projection of a profinite domain that has a different root. In particular, there cannot be a projection universal ω-profinite domain. We prove the next best thing: for each finite poset $A \cong \mathrm{rt}(A)$, there is a countable Plotkin poset V_A such that, if B is a countable Plotkin poset with $\mathrm{rt}(B) \cong A$, then $B \underset{\sim}{\triangleleft} V_A$. A fairly detailed outline of one technique of construction is offered here and we mention a second (closely related) technique. Kamimura and Tang [7] use a different approach to get a retraction universal model for the ω-profinite domains having bottoms. Their model, like $\mathbf{P}\omega$ and $\mathbb{T}^{\omega}$, is locally finite but is somewhat less natural than either of those models. In the opinion of the author, however, the construction described below does the most to reveal the fundamental *idea* that gives the existence result and yields the most detailed description of the model being built. (We are even able to draw a partial picture of it!) We begin by stating an interesting structure theorem for Plotkin posets.

Proposition 21 *If A and B are finite posets such that $A \triangleleft B$ but $A \neq B$, then there are posets $A_0, \ldots, A_n$ such that*

$$A = A_0 \triangleleft A_1 \triangleleft \cdots \triangleleft A_{n-1} \triangleleft A_n = B$$

and, for each $k < n$, $A_{k+1} - A_k$ is a singleton.

Proof. If $B - A$ is a singleton then we are done. Assume that the result holds for any pair $A' \lhd B'$ such that $B' - A'$ has fewer that n elements where $n > 1$. Suppose there are n elements in $B - A$ and let X be a maximal element of $B - A$, *i.e.* if $Y \in B$ such that $X \sqsubset Y$, then $Y \in A$. Set $A' = A \cup \{X\}$. We show that $A' \lhd B$. Let $Z \in B$ and suppose $u = \{Y \in A' \mid Y \sqsubseteq Z\}$. We must demonstrate that u has a largest element. If $u \subseteq A$ then this follows from the fact that $A \lhd B$. If $X \in u$ then $X \sqsubseteq Z$ so $X = Z$ or $Z \in A$. In either case, Z is the largest element of u. Hence $A' \lhd B$. Since $A \lhd B$ we have $A \lhd A' \lhd B$. But $B - A'$ has $n - 1$ elements, so by the induction hypothesis, there are posets $A_1, \ldots, A_n$ such that $A \lhd A' = A_1 \lhd \cdots \lhd A_n = B$. □

Theorem 22 *(Enumeration) If A is a countable Plotkin poset and $B = \mathrm{rt}(A)$, then there is an enumeration $X_0, X_1, \ldots$ of A such that for each n, $B \cup \{X_i \mid i < n\} \lhd A$.*

Proof. Suppose $\mathrm{rt}(A) = A_0 \lhd A_1 \lhd \cdots$ is a chain of finite normal substructures of A such that $A = \bigcup_{n \in \omega} A_n$. Let $B_0 \lhd B_1 \lhd \cdots$ be a new chain that results from deleting A_{n+1} for each n if it equals A_n. Using Lemma 21 we may refine this chain to a chain $C_0 \lhd C_1 \lhd \cdots$ such that $C_0 = \mathrm{rt}(A)$ and, for each n, $C_{n+1} - C_n$ is a singleton Z_n. Now, let $X_0, \ldots, X_{k-1}$ be an enumeration of C_0 and let $X_{n+k} = Z_n$ for each n. This enumeration has the desired property. □

Definition: Let $\langle A, \sqsubseteq \rangle$ be a poset. For each $X \in A$, let $\mathbf{X}$ be a constant symbol naming X. Let $\preceq$ be a binary relation symbol which is interpreted by $\sqsubseteq$. A *diagram type* over A is a set Γ of inequalities and negations of inequalities between constant symbols and a variable $\mathbf{v}$, *i.e.* formulas of the form

$$\mathbf{v} \preceq \mathbf{X}, \qquad \mathbf{v} \not\preceq \mathbf{X}, \qquad \mathbf{X} \preceq \mathbf{v}, \qquad \mathbf{X} \not\preceq \mathbf{v}$$

where $X \in A$. If $\langle A, \sqsubseteq \rangle$ is a substructure of $\langle B, \sqsubseteq \rangle$ and $Z \in B$, then the *diagram type of Z over A* is the set of all such equations (using constant symbols for elements of A) that hold when $\mathbf{v}$ is given the value Z and $\preceq$ is interpreted as the order relation on B. A diagram type Γ over A is said to be *realized* in B by Z if Γ is a subset of the diagram type of Z over A. A diagram type Γ over a poset A is said to be *normal* if there is a poset B with $A \lhd B$ such that Γ is realized in B. □

Lemma 23 *If Γ is a normal type over a finite poset B and A is a finite poset with $B \lhd A$, then there is a finite poset A_1 such that $A \lhd A_1$ and Γ is realized by some $Z \in A_1$ such that $B \cup \{Z\} \lhd A_1$.*

Proof. Let $\sqsubseteq$ be the partial ordering on A. Since $B \lhd A$, B inherits this ordering. Suppose $B \lhd A_0$ and $Z \in A_0$ such that Z realizes Γ. Let $\sqsubseteq_0$ be the partial ordering on A_0. Note that the restriction of $\sqsubseteq_0$ to B is the same as the restriction of $\sqsubseteq$ to B. Let $A_1 = A \cup \{Z\}$ and define a binary relation $\sqsubseteq_1$ on A_1 as follows:

- $Z \sqsubseteq_1 Z$,
- if $X, Y \in A$ then $X \sqsubseteq_1 Y$ iff $X \sqsubseteq Y$,
- if $X \in A$ then $X \sqsubseteq_1 Z$ iff there is an $X' \in B$ such that $X \sqsubseteq X' \sqsubseteq_0 Z$,
- if $X \in A$ then $Z \sqsubseteq_1 X$ iff there is an $X' \in B$ such that $Z \sqsubseteq_0 X' \sqsubseteq X$.

To see that $\langle A_1, \sqsubseteq_1 \rangle$ is a poset, note first of all that $\sqsubseteq_1$ is the transitive closure of $(\sqsubseteq \cup \sqsubseteq_0) \cap (A_1 \times A_1)$. That $\sqsubseteq_1$ is reflexive is immediate from its definition. To see that it is anti-symmetric, suppose $X \sqsubseteq_1 Z \sqsubseteq_1 X$ for some $X \in A$. Then there are $X_0, X_1 \in B$ such that $X \sqsubseteq X_0 \sqsubseteq_0 Z$ and $Z \sqsubseteq_0 X_1 \sqsubseteq X$. But then $X \sqsubseteq X_0 \sqsubseteq X_1 \sqsubseteq X$, so $X_0 = X_1 = X$ and therefore $X \in B$. Hence $X \sqsubseteq_0 Z \sqsubseteq_0 X$ implies $X = Z$ by the anti-symmetry of $\sqsubseteq_0$. Of course, if $X, Y \in A$ and $X \sqsubseteq_1 Y \sqsubseteq_1 X$, then $X = Y$ since $X \sqsubseteq Y \sqsubseteq X$.

Now, the fact that A is a substructure of A_1 is built into the definition of $\sqsubseteq_1$. To see that $A \lhd A_1$, suppose $u \subseteq A$ is finite and $u \sqsubseteq_1 Z$. By the definition of $\sqsubseteq_1$, for each $X \in u$ there is an $X' \in B$ such that $X \sqsubseteq X' \sqsubseteq_1 Z$. So let $u' = \{X' \mid X \in u\}$. Then $u' \sqsubseteq Z$. Since $B \lhd A_0$, there is a $Z' \in B$ such that $u' \sqsubseteq_0 Z' \sqsubseteq_0 Z$. But this implies that $u \sqsubseteq_1 Z' \sqsubseteq_1 Z$, so we may infer that $A \lhd A_1$. We must show that $B \cup \{Z\} \lhd A_1$. Suppose $u \subseteq B \cup \{Z\}$ is finite and $u \sqsubseteq_1 X$ for some $X \in A_1$. We must find a $Y \in B \cup \{Z\}$ such that $u \sqsubseteq_1 Y \sqsubseteq_1 X$. If $X = Z$ then the result is immediate—just let $Y = X$. So suppose $X \in A$. If $Z \notin u$ then we can get the desired Y by using the fact that $B \lhd A$. If $Z \in u$ then there is an $X' \in B$ such that $Z \sqsubseteq_0 X' \sqsubseteq X$. Thus

$$v = (u - \{Z\}) \cup \{X'\} \sqsubseteq X.$$

Since $B \lhd A$ and $v \subseteq B$, there is some $Y \in B$ such that $v \sqsubseteq Y \sqsubseteq X$. Since $Z \sqsubseteq_0 X' \sqsubseteq Y$ we may conclude that $Z \sqsubseteq_1 Y$. Thus $u \sqsubseteq_1 Y$ and we are done.

Finally, suppose $\mathbf{v} \preceq \mathbf{X}$ is in Γ for some $X \in B$. Then $Z \sqsubseteq_0 X$ since Z realizes Γ in A_0. Hence, by definition, $Z \sqsubseteq_1 X$. Suppose $\mathbf{v} \not\preceq \mathbf{X}$ is in Γ but $Z \sqsubseteq_1 X$. Then $Z \sqsubseteq_0 X$. But this contradicts the assumption that Z realizes Γ in A_0. So apparently $Z \not\sqsubseteq_1 X$. Similarly, the other formulas in Γ must be realized by Z in A_1. □

Lemma 24 *Let A be a finite poset. Then there is a finite poset A^+ such that $A \lhd A^+$ and, for every substructure $B \lhd A$ and normal type Γ over B, there is a $Z \in A^+$ such that Z realizes Γ and $B \cup \{Z\} \lhd A^+$.*

Proof. Let $\Gamma_1, \ldots, \Gamma_n$ be all of the normal types over normal substructures of A. Set $A = A_0$ and suppose $A \triangleleft A_k$. Suppose Γ_{k+1} is normal over $B \triangleleft A$. Then $B \triangleleft A_k$ so, by Lemma 23 there is a finite poset A_{k+1} such that $A_k \triangleleft A_{k+1}$ and $B \cup \{Z\} \triangleleft A_{k+1}$ for some Z that realizes Γ_{k+1}. Set $A^+ = A_{n+1}$. If Z realizes Γ_{k+1} in A_{k+1} then it realizes it also in A^+. Moreover, $B \cup \{Z\} \triangleleft A_{k+1} \triangleleft A^+$. □

Theorem 25 *Let V be a countable Plotkin poset. Suppose that for every finite $A \triangleleft V$ and normal type Γ over A, there is a realization Z for Γ such that $A \cup \{Z\} \triangleleft V$. If B is a countable Plotkin order such that $\mathrm{rt}(B) \cong \mathrm{rt}(V)$ then $B \underset{\sim}{\triangleleft} V$.*

Proof. Suppose B is a countable Plotkin order such that $\mathrm{rt}(B) \cong \mathrm{rt}(V)$. We may assume that B is a poset. By the Enumeration Theorem, there is an enumeration $X_0, X_1, \ldots$ of B such that for each $n \in \omega$,

$$B_n = \mathrm{rt}(B) \cup \{X_i \mid i < n\} \triangleleft B.$$

Since $B_0 = \mathrm{rt}(B)$, there is an isomorphism $f_0 : B_0 \cong V_0$ where $V_0 = \mathrm{rt}(V)$. We construct an ω-sequence of isomorphisms $f_n : A_n \cong V_n$ where $V_n \triangleleft V$, $f_n \subseteq f_{n+1}$ and $V_n \subseteq V_{n+1}$.

Suppose that f_n and V_n are given. Now, $B_n \triangleleft B_{n+1}$ so the diagram type Γ of X_n over B_n must be normal. Let Σ be the corresponding type over V_n, *i.e.* Σ is obtained from Γ by replacing any occurrence of a constant symbol for an $X \in A_n$ by a constant symbol for $f_n(X)$. Then Σ is a normal type over V_n so, by the hypothesis on V, there is a realization $Y_n \in V$ of Σ such that

$$V_{n+1} = V_n \cup \{Y_n\} \triangleleft V.$$

If we define $f_{n+1} : A_{n+1} \to V_{n+1}$ by

$$f_{n+1}(X) = \begin{cases} f_n(x) & \text{if } X \in A_n; \\ Y_n & \text{if } X = X_n, \end{cases}$$

then $f_n \subseteq f_{n+1}$ and f_{n+1} is an isomorphism. If $f = \bigcup_{n \in \omega} f_n$ and $V' = \bigcup_{n \in \omega} V_n$ then $f : B \cong V'$. Moreover, since $V_n \triangleleft V$ for each n, $V' \triangleleft V$. Hence $B \underset{\sim}{\triangleleft} V$. □

Corollary 26 *Let A be a finite poset such that $A \cong \mathrm{rt}(A)$. There is a Plotkin poset V_A such that, whenever B is a countable Plotkin order with $\mathrm{rt}(B) \cong A$, then $B \underset{\sim}{\triangleleft} V_A$.*

Proof. Let $A = A_0$ and, for each n, define $A_{n+1} = A_n^+$. Let $V_A = \bigcup_{n \in \omega} A_n$. Suppose $C \triangleleft V_A$ is finite. Then $C \triangleleft A_n$ for some n. If Γ is a normal type over C then Γ is realized by a $Z \in A_n^+ = A_{n+1}$ such that $C \cup \{Z\} \triangleleft A_{n+1}$. Since $A_{n+1} \triangleleft V_A$, the hypotheses of Theorem 25 are satisfied and the desired conclusion therefore follows. □

It is possible to get the A^+ in Lemma 24 by explicit construction. One way to do this is to pre-order the set $A_{tp} = \{\Gamma \mid \Gamma \text{ is normal over some finite } B \triangleleft A\}$ by letting $\Gamma \vdash \Sigma$ just

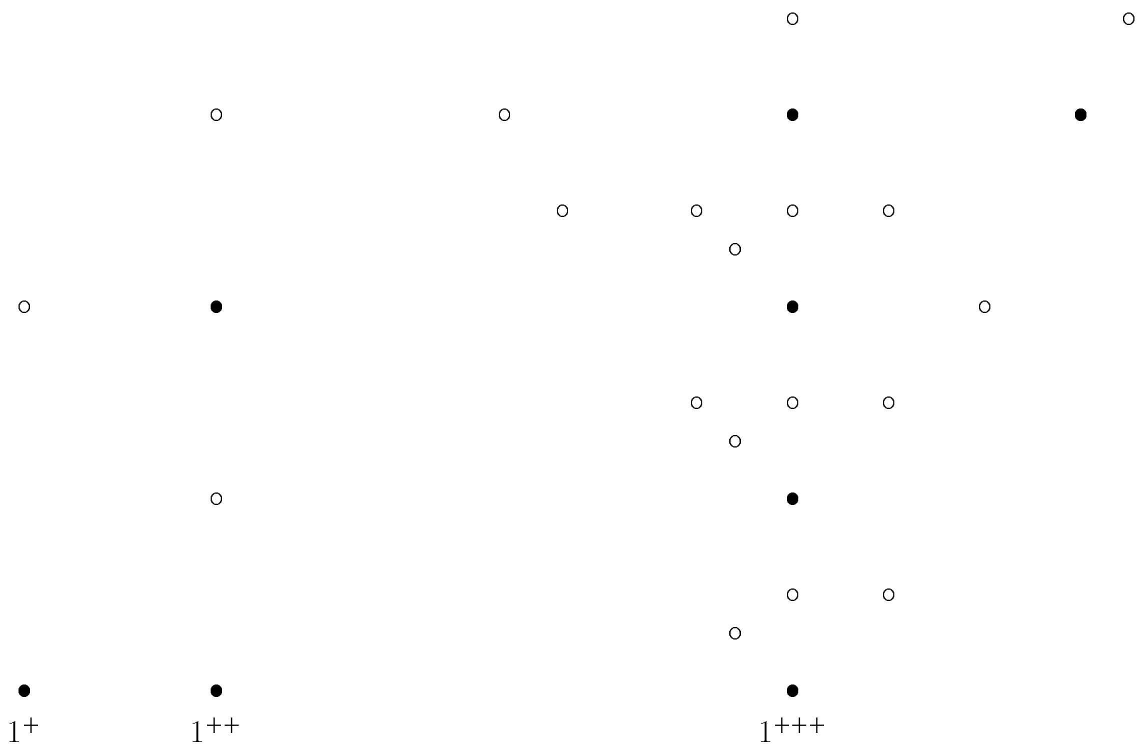


Figure 3: Construction of V_1.

in case there are $X, Y \in A$ such that $\mathbf{v} \preceq \mathbf{X}$ is in Γ, $\mathbf{Y} \preceq \mathbf{v}$ is in Σ, and $X \sqsubseteq Y$. If we let $A^+ = \tilde{A}_{tp}$ then there is a normal substructure $A' \triangleleft A^+$ with $A \cong A'$ such that, for every normal type Γ over a substructure $B \triangleleft A'$, there is a $Z \in A^+$ such that $B \cup \{Z\} \triangleleft A^+$ and Z realizes Γ. To get a universal domain one solves the domain equation $A = A^+$. There is an even more explicit way of describing this operation which was remarked to the author by Dana Scott. Given a finite poset A, let A^+ be the set of pairs $\langle X, u\rangle$ such that $X \in A$ and u is an upwards closed set of points from A such that $X \sqsubseteq Y$ for each $Y \in u$. Say that $\langle X, u\rangle \sqsubseteq \langle Y, v\rangle$ iff $Y \in u$. This more order-theoretic way of doing things helps in picturing the universal domain as the limit of the posets $A \trianglelefteq A^+ \trianglelefteq A^{++} \trianglelefteq \cdots$. Figure 3 illustrates the first three stages in the construction of the universal domain V_1 with a trivial root.

6 Join Completion.

In this section we present the *join completion* operator J. For a pre-order A,

$$J(A) = \{u \subseteq A \mid u \text{ is finite and bounded}\}$$

and if $u, v \in J(A)$, then

$$u \vdash_{J(A)} v \text{ iff } \forall X \in A.\ X \vdash_A u \Rightarrow X \vdash_A v.$$

The following proposition lists some of the properties of J:

Theorem 27 *Let A and B be pre-orders. Then*

1. *$\langle J(A), \vdash_{J(A)}\rangle$ is bounded complete;*
2. *if A is bounded complete then $J(A) \cong A$;*
3. *J is continuous;*
4. *$J(A \times B) \cong J(A) \times J(B)$.*

Proof. (1) Suppose $u, v \in J(A)$ and $w \vdash_{J(A)} u, v$. Then $u \cup v$ is bounded in A by anything that bounds w. Hence $u \cup v$ is in $J(A)$ and $w \vdash_{J(A)} u \cup v$. But any bound for $u \cup v$ in A is a bound for u and a bound for v, so $u \cup v \vdash_{J(A)} u, v$. Thus $J(A)$ has bounded joins.

(2) Suppose A is bounded complete and define $f \subseteq A \times J(A)$ by $X\ f\ u$ if and only if $X \vdash_A u$. To see that f is approximable, just note that $X\ f\ u$ if and only if $X \vdash_A Y$ where Y is a join for u. Hence, if $X\ f\ u, v$ then $X \vdash_A Y$ where Y is the join of $u \cup v$ so $X\ f\ u \cup v \vdash_{J(A)} u, v$. The other conditions for approximability of f are obviously satisfied. Define $g \subseteq J(A) \times A$ by $u\ g\ X$ if and only if $u \vdash_{J(A)} \{X\}$. If $u\ g\ X$ and $u\ g\ Y$, then

$u\ g\ Z$ where Z is a join for u. The remaining condition for approximability of g is obviously satisfied. Now, suppose $X\ f\ u$ and $u\ g\ Z$ for some $X, Z \in A$ and $u \in J(A)$. If Y is a join for u, then $X \vdash_A Y \vdash_A Z$ so $X \vdash_A Z$. Therefore $g \circ f \subseteq \mathrm{id}_A$. If, on the other hand, $X \vdash_A Z$ then $X\ f\ \{X\}\ g\ Z$ so $g \circ f \supseteq \mathrm{id}_A$. Hence $g \circ f = \mathrm{id}_A$. Now, suppose $u\ g\ X$ and $X\ f\ w$ for some $u, w \in J(A)$ and $X \in A$. Then $u \vdash_{J(A)} \{X\}$ and $X \vdash_A Y$ where Y is a join of w. Hence $\{X\} \vdash_{J(A)} \{Y\} \vdash_{J(A)} w$ so $u \vdash_{J(A)} w$. Therefore $f \circ g \subseteq \mathrm{id}_{J(A)}$. If, on the other hand, $u \vdash_{J(A)} w$ then $u \vdash_{J(A)} \{Y\}$ for a join Y of w. Thus $u\ g\ Y$ and $Y\ f\ w$ so $f \circ g \supseteq \mathrm{id}_{J(A)}$. Hence $f \circ g = \mathrm{id}_{J(A)}$.

(3) We must first show that if $A \triangleleft B$ then $J(A) \triangleleft J(B)$. Suppose $A \triangleleft B$. If u is bounded in A, then it is bounded in B, so any element of $J(A)$ is also an element of $J(B)$. Suppose $u, v \in J(A)$ and $u \vdash_{J(A)} v$. We claim that $u \vdash_{J(B)} v$. Suppose $X \in B$ and $X \vdash_B u$. Since $A \triangleleft B$, there is an $X' \in A$ such that $X \vdash_B X' \vdash_A u$. But $u \vdash_{J(A)} v$ means $X' \vdash_A v$. Hence $X \vdash_B v$ and the claim is established. Obviously, $u \vdash_{J(B)} v$ implies $u \vdash_{J(A)} v$. Thus $\langle J(A), \vdash_{J(A)} \rangle \subseteq \langle J(B), \vdash_{J(B)} \rangle$. To see that $J(A) \triangleleft J(B)$, suppose $u, v \in J(A)$ and $w \vdash_{J(B)} u, v$ for some $w \in J(B)$. If $X \vdash_A w$ for some $X \in B$, then $X \vdash_A u \cup v$; so $u \cup v$ is bounded and there is an $X' \in A$ such that $X' \vdash_A u \cup v$. Hence $u \cup v \in J(A)$ and we conclude that $J(A)$ is closed under existing joins in $J(B)$. Thus $J(A) \triangleleft J(B)$. To see that J is continuous, suppose $B = \bigcup \mathcal{M}$, where $\mathcal{M}$ is a directed collection of normal substructures of B. If $u \in J(B)$ then $u \subseteq A$ for some $A \in \mathcal{M}$ so $u \in J(A)$. Hence $J(B) \subseteq \bigcup_{A \in \mathcal{M}} J(A)$. The opposite inclusion is obvious.

(4) Left for the reader. □

By Corollary 26, there is a Plotkin order V_1 such that, whenever A is a Plotkin order with a least element, we have $A \trianglelefteq V_1$. We may extract from Theorem 27 the following:

Corollary 28 *If A is a bounded complete pre-order then $A \trianglelefteq J(V_1)$.*

Proof. Since A has a least element we know that $A \cong A'$ for some $A' \triangleleft V_1$. But A' is bounded complete, so $A' \cong J(A')$. Hence $A \cong J(A') \triangleleft J(V_1)$. □

Now, suppose u and v are finite bounded subsets of V_1 such that $u, v \neq \{\bot\}$. Consider the diagram type

$$\Gamma(\mathbf{v}) = \{\bot \neq \mathbf{v}\} \cup \{\mathbf{v} \sqsubseteq \mathbf{X} \mid X \in u \cup v\}.$$

This type is normal over $\mathcal{U}^*_{V_1}(u \cup v)$ so it has a realization Z in V_1. But $u \vdash_{J(V_1)} \{Z\}$ and $v \vdash_{J(V_1)} \{Z\}$ and $\{Z\} \not\sim \{\bot\}$. This shows that no pair $u, v \neq \{\bot\}$ of bounded subsets of $J(V_1)$ can be complementary to one another. Hence $J(V_1)$ cannot be isomorphic to $\mathbb{B}^-$. We conclude that, although $|J(V_1)|$ is projection universal for bounded complete algebraic dcpo's, it is not isomorphic to Scott's universal domain $\mathbb{U}$.

A variant on the join completion operator has been studied independently in [5] for a different purpose. The Frink completion $\|A\|$ of a pre-order A is defined there. This operation is related to the join completion by the isomorphism $\|A\| \cong |J(A)^{\top}|$ were $(\cdot)^{\top}$ is the operation that adds a new greatest element $\top$.

7 Fixed Points of Continuous Operators.

In this last section we prove a theorem which gives the conditions under which a domain equation involving continuous operators has a profinite solution. Solutions to such equations over the profinite domains are more problematic than is the case for strongly algebriac domains or bounded complete algebraic domains. In these latter categories, *every* such equation has a solution. This is not true for the profinites because there is no terminal object in the category of profinite domains and projections. That is, there is no profinite domain T such that, for every profinite D, there is a $\langle p, q \rangle : D \xrightarrow{\text{pe}} T$. The single element poset 1 will not suffice, because it cannot be embedded in 1+1 for example. The following theorem provides a reasonably simple existence condition:

Theorem 29 *Suppose* $F : \omega\text{-}\mathbf{PLT} \to \omega\text{-}\mathbf{PLT}$ *is continuous. Then* F *has a fixed point in* ω-**PLT** *whose root is isomorphic to a poset* A *if and only if* $A \cong \mathrm{rt}(F(A))$.

Proof. To prove necessity ($\Rightarrow$), suppose $F(B) \cong B$ for a Plotkin order B. Then $\mathrm{rt}(B) \cong \mathrm{rt}(F(B))$. But $\mathrm{rt}(B) \triangleleft B$ so $F(\mathrm{rt}(B)) \triangleleft F(B)$ by monotonicity of F. Hence $\mathrm{rt}(F(\mathrm{rt}(B))) = \mathrm{rt}(F(B))$ and therefore $\mathrm{rt}(B) \cong \mathrm{rt}(F(\mathrm{rt}(B)))$. If $A \cong \mathrm{rt}(B)$ then $A \cong \mathrm{rt}(F(A))$. To prove sufficiency ($\Leftarrow$), suppose $A \cong \mathrm{rt}(F(A))$. Then by Theorem 26 there is a pre-order V_A and a map $i : F(A) \trianglelefteq V_A$. Consider the function $N(i) \circ F : N(V_A) \to N(V_A)$. By Lemma 20 this function is continuous so, by the Tarski Fixed Point Theorem, it has a least fixed point B. So $B = N(i)(F(B)) \cong F(B)$. □

We now discuss the application of Theorem 29 to some specific equations. It is possible to show that, for any pair of pre-orders A and B having property m, $\mathrm{rt}(A \times B) = \mathrm{rt}(A) \times \mathrm{rt}(B)$. In light of Theorem 29 this is noteworty in the following regard. Since the product is continuous, the operator $F(A) = A \times A$ is continuous. Since $F(A)$ is finite whenever A is, F cuts down to a continuous operator on ω-**PLT**. Suppose $A \cong F(A)$ is a Plotkin order and let $B = \mathrm{rt}(A)$. Now, B is finite so suppose it has m elements. Then $\mathrm{rt}(F(A)) = \mathrm{rt}(A \times A) = B \times B$ has m^2 elements. Since $\mathrm{rt}(A) \cong \mathrm{rt}(F(A))$ we must have $m = m^2$ so apparently $m = 1$ or $m = 0$. In other words, a non-empty fixed point in ω-**PLT** of the equation $A \cong F(A)$ must have a least element. This result carries over to the ω-profinite domains, because an ω-profinite solution of the equation $D \cong D \times D$ gives rise to the solution $\mathbf{B}_D \cong \mathbf{B}_D \times \mathbf{B}_D$ in

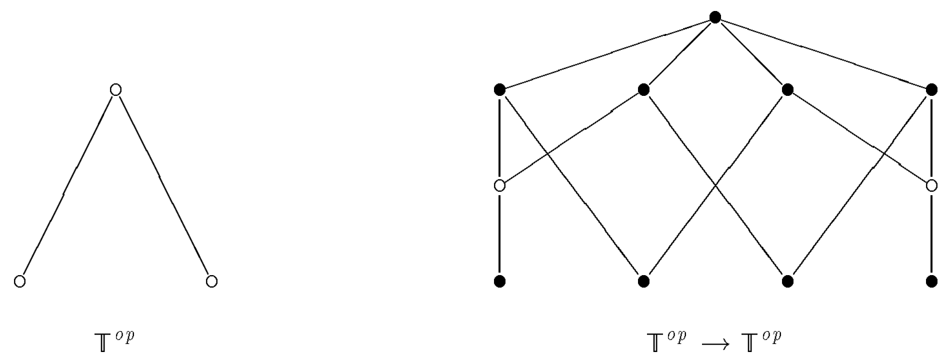


Figure 4: Root of a function space.

ω-**PLT**. A similar situation occurs with the diagonal of the coproduct. One can show that if A and B have property m, then $\mathrm{rt}(A+B) = \mathrm{rt}(A) + \mathrm{rt}(B)$. Hence the only ω-profinite solution to the equation $D \cong D + D$ is the initial object 0.

The diagonal of the function space operator, $F(A) = A^A$, is more interesting. It is *not* true, in general, that $\mathrm{rt}(B^A) \cong \mathrm{rt}(B)^{\mathrm{rt}(A)}$. Consider, for example, the opposite $\mathbb{T}^{op}$ of the truth value dcpo. The monotone functions from $\mathbb{T}^{op}$ into $\mathbb{T}^{op}$ form a poset whose root is not isomorphic to the poset $\mathrm{rt}(\mathbb{T}^{op}) \to \mathrm{rt}(\mathbb{T}^{op}) = \mathbb{T}^{op} \to \mathbb{T}^{op}$. Hasse diagrams for $\mathbb{T}^{op}$ and $\mathbb{T}^{op} \to \mathbb{T}^{op}$ appear in Figure 4. The root of $\mathbb{T}^{op} \to \mathbb{T}^{op}$ is drawn in black there.

Suppose A is a non-empty finite poset and $A \cong \mathrm{rt}(A^A)$. We claim that A is isomorphic to the trivial one element poset. To see this, suppose A is not isomorphic to 0 or 1. We may assume that A is a poset; since A is finite, A^A is isomorphic to the poset $A \to A$ of monotone functions from A into A. Now, A has a set of n minimal elements where $n > 1$. A constant function mapping all of A to a minimal element of A is minimal in $A \to A$ so $\mathrm{rt}(A \to A)$ has a least n minimal elements. Let $f : A \to A$ be monotone and suppose f is below the identity function on A. Suppose $X \in A$ and $f(Y) = Y$ for every $Y \sqsubset X$. Using the fact that A has no proper normal substructure, one can show that there is a set $u \subseteq A$ such that X is a minimal upper bound of u. But then $u = f(u) \sqsubseteq f(X) \sqsubseteq X$ so $f(X) = X$. Hence f is the identity function and consequently the identity function is minimal in $A \to A$. Since none of the constant functions is equal to the identity function, this means $\mathrm{rt}(A \to A)$ has at least $n+1$ minimal elements. Hence, we cannot have $A \cong \mathrm{rt}(A \to A)$. This shows that a non-empty fixed point of the operator F in ω-**PLT** must have a least element. Again, this can be used to show that if $D \cong D \to D$ is ω-profinite and non-empty, then D has a least element.

We conclude with short notes on powerdomains and models of λ-calculus. The convex powerdomain was introduced by Plotkin [17]. Smyth [26] introduced the upper powerdomain and gave a detailed description of Plotkin's powerdomain and his using the finite elements of the domains. The definition below, which appears in Scott [25], describes these operators and the lower powerdomain through their action on pre-orders. The names for the operators are derived from mathematical considerations [27].

Let A be a pre-order and suppose M_A is the set of finite non-empty subsets of A. The *upper powerdomain* $A^\sharp$ of A is the set M_A together with a pre-ordering $\vdash_\sharp$ given by

$$u \vdash_\sharp v \text{ iff } (\forall X \in u)(\exists Y \in v).\ X \vdash Y.$$

Dually, the *lower powerdomain* $A^\flat$ of A is M_A with the pre-ordering $\vdash_\flat$ given by

$$u \vdash_\flat v \text{ iff } (\forall Y \in v)(\exists X \in u).\ X \vdash Y.$$

The *convex powerdomain* $A^\natural$ of A is the intersection of the upper and lower powerdomain pre-orderings on M_A, *i.e.*

$$u \vdash_\natural v \text{ iff } u \vdash_\sharp v \text{ and } u \vdash_\flat v.$$

If $f : A \to B$ is approximable then we define

$$u\ f^\sharp\ v \text{ iff } (\forall X \in u)(\exists Y \in v).\ X\ f\ Y$$
$$u\ f^\flat\ v \text{ iff } (\forall Y \in v)(\exists X \in u).\ X\ f\ Y$$
$$u\ f^\natural\ v \text{ iff } u\ f^\sharp\ v \text{ and } u\ f^\flat\ v.\ \Box$$

For each of the pre-orders $A^\sharp$, $A^\flat$ and $A^\natural$, it is possible to define a binary operator which acts like a union function. For example, $\text{union}^\natural : A^\natural \times A^\natural \to A^\natural$ is the approximable relation given by defining

$$(u, v)\ \text{union}^\natural\ w \text{ iff } u \cup v \vdash_\natural w.$$

There is also a singleton relation $\text{singleton}^\natural : A \to A^\natural$ given by

$$X\ \text{singleton}^\natural\ u \text{ iff } \{X\} \vdash_\natural u.$$

It is straight-forward to show that the operators $(\cdot)^\sharp$, $(\cdot)^\flat$ and $(\cdot)^\natural$ are continuous. Since each of them obviously sends finite posets to finite posets, Theorem 10 shows that they are closed on **PLT**. It is well-known that the convex powerdomain does not preserve the property of bounded completeness (look in [17] for a counterexample). It is not closed over any of the first three classes listed in Table 1. In fact, it is rather difficult to find a cartesian closed subcategory of **PO** which *is* closed under $(\cdot)^\natural$. **PLT** and some slight variants (such as the Plotkin orders having bottoms) are the only known examples. If one alters the definition of $(\cdot)^\natural$ by allowing M_A to include the emptyset, then the resulting operator does not even preserve the property of having a least element. Further discussion of the properties of these operators can be found in [16] and [15].

The precise relationship between the bounded complete algebraic dcpo's and the profinites is not well understood. Although the join completion operator does provide some connection, it does not seem to be useful in resolving some of the open questions. For example, it is not known (at least to the author) whether $A \mathrel{\underset{\sim}{\triangleleft}} (\mathbb{B}^-)^\natural$ for every countable Plotkin order A with a least element. As an aside: it is possible to show that $A \mathrel{\underset{\sim}{\triangleleft}} A^\natural$ for every bounded complete A. This fact makes it possible to find non-trivial solutions to the equation $A \cong A^\natural$.

As far as formal semantics goes, the poset V_1 is almost surely the most interesting of the posets V_A produced in Section 5. Since $V_1^{V_1}$ has a least element, we know that $V_1^{V_1} \mathrel{\underset{\sim}{\triangleleft}} V_1$. Hence $V_1^{V_1}$ is a retract of V_1. Since **PLT** is a concrete cartesian closed category, we may conclude that V_1 is a model of the *untyped* $\lambda\beta$*-calculus* (see [2] and [10]). But there is something more which is true. It is possible to prove that $N(V_1) \mathrel{\underset{\sim}{\triangleleft}} V_1^{V_1}$ (see [3]), so the theory described in [22] and [23] for $\mathbb{U}$ applies also to $\mathbb{V} = |V_1|$. In particular, $\mathbb{V}$ is a finitary projection model of the *polymorphic* λ*-calculus* in the sense of [1]. It seems unlikely that the theory of $\mathbb{V}$ is much different from that of $\mathbb{U}$, but it is a "bigger" model in the sense that there is a projection from $\mathbb{V}$ onto $\mathbb{U}$. Moreoever, the powerdomain operators mentioned above are definable on the types of $\mathbb{V}$, and this is not true of $\mathbb{U}$.

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