Coherence and Consistency in Domains*

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Abstract

Almost all of the categories normally used as a mathematical foundation for denotational semantics satisfy a condition known as *consistent completeness.* The goal of this paper is to explore the possibility of using a different condition—that of *coherence*—which has its origins in topology and logic. In particular, we concentrate on those posets whose principal ideals are algebraic lattices and whose topologies are coherent. These form a cartesian closed category which has fixed points for domain equations. It is shown that a "universal domain" exists. Since the construction of this domain seems to be of general significance, a categorical treatment is provided and applied to other classes of domains. Universal domains constructed in this fashion enjoy an additional property: they are *saturated.* We show that there is exactly one such domain in each of the classes under consideration.

1 Introduction.

The first structures used as a mathematical foundation for the denotational semantics of programming languages were lattices. With lattices it was possible to solve the necessary recursive equations and an elegant mathematical theory could be developed using the familiar category of (countably based) algebraic lattices [Sco76] (although it was necessary to take some care to choose the right notion of morphism). As experience with denotational semantics grew, deeper computational intuitions were developed and new categories were introduced in attempts to match these intuitions to the mathematical constructs. For example, it was desirable to have a class of domains which included such structures as the partial functions from natural numbers to natural numbers which—under their usual ordering—do not form a lattice. Such theories were proposed by Plotkin [Plo78], Berry [Ber78] and also Scott [Sco81, Sco82a, Sco82b].

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The category which Scott proposed was very similar to the algebraic lattices: a dcpo D is said to be a *Scott domain* (or bounded complete domain) if the dcpo $D^{\top}$ obtained by adding a top to D is an algebraic lattice (with a countable basis). The arrows of the category are continuous functions, *i.e.* monotone functions which preserve joins of directed collections of elements. The category of Scott domains is easy to work with and has an intuitive logical character which has been the subject of several investigations (see, in particular, [Sco82a, Abr87]). One central feature of these treatments is the concept of *consistency* of data. One may think of a Scott domain as a collection of propositions or data elements under an ordering of *partial information.* An element x is ordered below an element y in a domain D if x is "more partial" than y. The element x is a kind of partial description of y. Now, given two data elements x_1 and x_2, there may or may not be a third element y which they describe. If there is such a y, then x_1 and x_2 are said to be *consistent,* otherwise they are *inconsistent.* A crucial feature of a Scott domain is the following fact: *if two elements of a Scott domain D are consistent, then they have a join in D.* This property is commonly referred to as *consistent completeness.*

The use of consistent complete domains for modeling the semantics of types in programming languages has become the general practice. However, we would like to note in this paper that *it is not the only reasonable direction the theory could have taken* at the point that consistency was recognized as a central concept. Up until the time we are writing this paper, almost all of the categories of domains that have been proposed as a possible foundation for the semantics of programming languages have been (essentially equivalent to) dcpo's which satisfy the consistent completeness condition. This includes those categories which use stable continuous functions [Ber78, Gir86] as well as categories related to the Scott domains (such as the continuous lattices).[1] The one noteworthy exception is the category of ω-bifinite domains which was introduced by Plotkin [Plo76] (where it is called **SFP**). These will be discussed below.

One might apply the following line of reasoning in an attempt to deal with the concept of consistency of data. A domain is a collection of propositions providing partial descriptions of elements (which may also be propositions describing further elements); a given element dominates a collection of data elements which provide partial descriptions of it. We propose the following condition on the structure of the partial descriptions of an element: *the partial descriptions of an element must form an algebraic lattice.* Let us refer to this condition as *local algebraicity.* But a locally algebraic dcpo (with a countable basis) is just a Scott domain right? No, not at all! Aside from the fact that such a domain need not have a least element (an infinite discrete domain is locally algebraic for example) it is even possible that a consistent pair of elements have no join! (See Figure 1.) One can show, however, that almost all of the essential features needed to provide semantics for programming languages are satisfied by locally algebraic domains.

The concept of a locally algebraic domain was formulated by the second author who came across

[1] We omit from discussion categories of dcpo's with no assumptions about the existence of a basis.

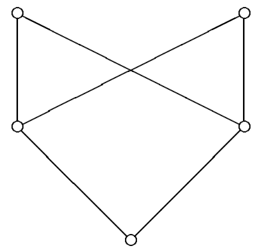

Figure 1: A locally algebraic domain which is not consistent complete.

the concept in the course of his investigations into extensions of Smyth's Theorem [Jun88b, Jun88a]. We refer to locally algebraic domains as *L-domains* to keep the terminology short. They were independently discovered by Thierry Coquand as a special instance of his categories of embeddings [Coq88]. We will discuss some basic properties of L-domains in the next section—for a more detailed discussion, the reader can examine [Coq88, Jun88b, Jun88a]. The bulk of the paper will focus on the properties of a subcategory of the L-domains which were introduced in the first author's doctoral dissertation [Gun85]. The category which was investigated there (the objects were called *short* domains) consisted of those L-domains which were ω-bifinite. It was observed at that time that such domains formed a cartesian closed category in which one could solve recursive domain equations. However, we would like to demonstrate a further fact about them below. Namely, that *there is a "universal" domain in this category.*

Our construction is similar to that which appears in [Gun87] for the ω-bifinite domains, but a more subtle ordering is needed to make things work properly. We prove a lemma expressed in categorical terms which aids one in demonstrating the existence of a universal domain by demonstrating the existence of what we call a *finite relative saturation.* This lemma is sufficiently general that it applies not only to our construction of a universal ω-bifinite L-domain and the construction of a universal ω-bifinite domain as in [Gun87], but also to consistent complete domains and even countably based algebraic lattices! The universal domains so constructed are characterized by a property very similar to what model theories call *countable saturation* [CK73]. We prove that a model with this property is unique up to isomorphism. We can apply this result to show that Scott's universal domain for the consistent completes [Sco81, Sco82a, Sco82b] is *not* saturated.

The paper is divided into six sections which we overview briefly. Section two provides some definitions and establishes notation. A few basic propositions are also remarked. The third section discusses the coherence condition on the topology of a domain. We show how this condition translates into an order-theoretic one and discuss some important properties of domains with coherent topologies. The fourth section discusses the universal domain construction. Since this construction seems to have a general significance, we have attempted to provide a categorical treatment of it. This categorical treatment makes it possible to see the construction in this paper and the one that was presented in [Gun87] as instances of a more general theory which may have applications in other cases. In the fifth section we instantiate the general theory for the classes $\omega\mathbf{Lat}$ of algebraic lattices, $\omega\mathbf{S}$ of Scott domains, $\omega\mathbf{BL}$ of ω-bifinite L-domains and $\omega\mathbf{B}$ of ω-bifinite domains. The universal domains which we thus construct are saturated. We prove in Section 6 that any saturated object in a subclasss of $\omega\mathbf{B}^{ep}$ is universal and that there is at most one such object (up to isomorphism).

2 Basic definitions and facts.

For the purposes of this paper a dcpo (complete poset) is a poset $(D, \sqsubseteq)$ with least element and with joins $\bigsqcup M$ for all directed subsets M. A function $f\colon D \to E$ between dcpo's D and E is *continuous* if it is monotone and preserves joins of directed subsets of D. An element x of a dcpo D is said to be *compact* if, whenever M is a directed subset of D and $x \sqsubseteq \bigsqcup M$, then there is a $y \in M$ such that $x \sqsubseteq y$. Let $K(D)$ be the collection of compact elements of a dcpo D. A dcpo D is said to be *algebraic* if every element of D is the join of a directed collection of compact elements. D is said to be ω-algebraic if it is algebraic and $K(D)$ is countable. An *algebraic lattice* is an algebraic dcpo which is a lattice.

Definition: A dcpo D is *locally algebraic* if, for every $x \in D$, the principal ideal

$$\downarrow x = \{y \in D \mid y \sqsubseteq x\}$$

generated by x is an algebraic lattice. ▮

Proposition 1 *If D is locally algebraic, then it is algebraic.* ▮

Proof: Suppose c is a compact element in $\downarrow x$ and $(e_i)_{i\in I}$ is a directed collection of elements with supremum e above c. The principal ideal $\downarrow e$ is by assumption an algebraic dcpo, so in particular the element c is the supremum of a directed collection $(c_j)_{j\in J}$ of compact elements in the $\downarrow e$-sense. All these elements belong to $\downarrow x$ as well and since c is compact there, one of the elements c_j must be equal to c. Going back to $\downarrow e$ we learn that c is equal to a compact element in this ideal, so some e_i must be above c. This proves that any locally compact element is also globally compact and hence D is algebraic. ▮

To keep the terminology short, we will refer to locally algebraic dcpo's as *L-domains.* The category of L-domains properly contains the class of Scott-domains: Figure 1 shows an example. The difference between the two concepts is illustrated by the following characterizations:

Proposition 2 *Let D be an algebraic dcpo.*

- *D is a Scott-domain, if and only if every nonempty subset has a meet in D.*
- *D is an L-domain, if and only if every bounded nonempty subset has a meet in D.* ▮

(For a proof see [Jun88b].)

The difference may seem a slight one but it has some important consequences. The basis of the function space of a Scott-domain D has always the same cardinality as $K(D)$, whereas the cardinality may increase if D is an L-domain. However, the following (which was found independently by Thierry Coquand) remains true:

Theorem 3 *The category of L-domains and continuous functions is cartesian closed.* ▮

In [Jun88b] it is proved that, in the category of algebraic dcpo's with least element, there are exactly two maximal cartesian closed subcategories: the category of L-domains and the category of bifinite domains, which we now proceed to define.

A continuous function $f^L: D \to E$ between dcpo's D and E is said to be an *embedding* if there is a continuous function $f^R: E \to D$ such that $f^R \circ f^L = id_D$ and $f^L \circ f^R \sqsubseteq id_E$ where id_D and id_E are the identity functions on D and E respectively. If there is such a function f^R, then it is uniquely determined by f^L and is said to be the *projection* corresponding to f^L. Pairs $f = \langle f^L, f^R \rangle: D \to E$, where f^L is an embedding and f^R the corresponding projection, form the arrows of a category $\mathbf{DCPO}^{ep}$ which has dcpo's as its objects. Composition is given by

$$\langle f^L, f^R \rangle \circ \langle g^L, g^R \rangle = \langle f^L \circ g^L, g^R \circ f^R \rangle.$$

It is a basic fact in the theory of domains that $\mathbf{DCPO}^{ep}$ has directed colimits, which we call *bilimits* since they can be gotten either from the directed system of embeddings or from the codirected system of projections.

Theorem 4 *The category of L-domains and embedding-projection pairs has bilimits.* ▮

If a dcpo is a bilimit in $\mathbf{DCPO}^{ep}$ of a family of finite posets with least element, then it is said to be a *bifinite domain.* It is possible to show that bifinite domains must be algebraic. Let $\mathbf{B}$ and $\mathbf{B}^{ep}$ be the categories of bifinite domains with continuous functions and embedding-projection pairs respectively. It is possible to show that $\mathbf{B}$ is a cartesian closed category and $\mathbf{B}^{ep}$ has bilimits of directed families [Gun85, Gun87]. Bifinite domains with a countable basis and least element are the "**SFP**-objects" of Plotkin [Plo76]. We will follow Smyth's terminology [Smy83] and refer to them as *ω-bifinite* domains. We write $\omega\mathbf{B}$ for the category with continuous functions and $\omega\mathbf{B}^{ep}$ for the category with embedding-projection pairs. It is not hard to see that $\omega\mathbf{B}$ is a cartesian closed category and $\omega\mathbf{B}^{ep}$ has bilimits for countable directed families.

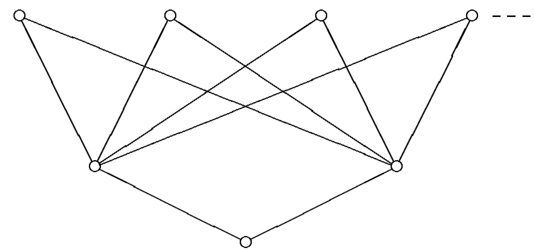

Figure 2: K has a countable basis, but $K \to K$ does not.

3 Coherence.

In order to get a satisfactory class of spaces as domains for denotational semantics it is desirable to impose a more restrictive condition than local algebraicity. Suppose one wished to define a notion of *computability* on L-domains. It might be possible to do this for the L-domains with a countable basis. So why not restrict oneself to these? The problem is that the L-domains with countable basis are not closed under the exponential! Consider the poset K pictured in Figure 2. This is an L-domain with a countable basis but $K \to K$ has a basis with continuum many members.

Since M. Smyth [Smy83] has proved that any domain which has an ω-algebraic function space is in fact bifinite, it is reasonable to investigate the category $\omega\mathbf{BL}$ of bifinite L-domains which have countable bases and least elements, *i.e.* the *ω-bifinite L-domains.* The poset in Figure 2 is a typical example of an L-domain that fails to be bifinite.

An unfortunate drawback to the bifiniteness condition is the fact that it is not very easy to understand. Although intrinsic descriptions are possible and these do help in reasoning about bifinite domains, it would still be nice to work with a simpler class of structures. However, it turns out that the ω-bifinite domains which are L-domains may be somewhat more easily characterized than ω-bifinite domains in general. In particular, they may be identified as those L-domains which have a "nice" Scott topology.

We will follow the definitions and notation in Johnstone [Joh82]. A dcpo D can be given a topology as follows. The open subsets of the topology are those which satisfy:

1. whenever $x \in U$ and $x \sqsubseteq y$, then $y \in U$, and
2. whenever $M \subseteq D$ is directed and $\bigsqcup M \in U$, then $M \cap U \neq \emptyset$.

This is usually called the *Scott topology* on D and it will be denoted ΣD. It is possible to show that a function $f\colon D \to E$ between dcpo's D and E is continuous in the sense that $f(\bigsqcup M) = \bigsqcup f(M)$, for any directed $M \subseteq D$, if and only if it is continuous in the usual topological sense—with respect to the Scott topology.

Definition: Let D be an algebraic dcpo. The topology ΣD is said to be *coherent* if the quasicompact open subsets of D are closed under finite intersections. ▮

We would like to make two brief remarks about this terminology. First, to keep things simple, we have restricted the definition to algebraic dcpo's; the definition above would not correspond to the usual notion of a coherent topology if D were allowed to be an arbitrary dcpo. Second, we would like to comment that the meaning for the term "coherent" which we have given should not be confused with other meanings from the domain theory literature. In particular, a poset is sometimes said to be coherent if any pairwise consistent set has a least upper bound. This condition is *stronger* than consistent completeness and certainly does not correspond to the condition we are using here!

Coherence is an elegant condition on the topology of a domain D which has an important significance for the order structure of D. Let us say that a poset P has the *strong minimal upper bounds property* (or *property M* for short) if, for every finite subset $A \subseteq P$, the set $mub(A)$ of minimal upper bounds of A satisfies the following properties:

1. $mub(A)$ has only finitely many elements and
2. $mub(A)$ is complete in the sense that for every $p \in P$, if $x \sqsubseteq p$ for every $x \in A$, then $y \sqsubseteq p$ for some $y \in mub(A)$.

We have the following:

Proposition 5 *Let D be an algebraic dcpo. Then ΣD is coherent if and only if the basis $K(D)$ of D has property M.* ▮

Proof: Since the sets of the form $\uparrow c$, with c a compact element of D, form a basis of the Scott topology, a set A is quasicompact open if and only if it is a finite union of such principal filters.

So let A and A' be upper sets generated by finite sets $M, M' \subseteq K(D)$, respectively. Each element of $A \cap A'$ is above some element of M and above some element of M'. So $A \cap A'$ is generated by the finite set $\bigcap_{m \in M, m' \in M'} mub(m, m')$ and hence itself quasicompact.

For the converse let $m \subseteq K(D)$ be a finite set. Each set $\uparrow m$, $m \in M$ is quasicompact open and, by coherence, so is $\bigcap_{m \in M} \uparrow m$. The latter set is therefore covered by finitely many principal open filters and hence generated by finitely many compact elements. This proves that $K(D)$ has property M. ▮

The central theorem of this section states that a bifinite L-domain may be characterized using the coherence condition:

Theorem 6 *Let D be an L-domain. Then ΣD is coherent if and only if D is bifinite.* ▮

Proof: It is well known (see [Plo76], for example) that the basis of a bifinite domain has property M, so by the previous proposition the 'only if'-part is taken care of.

For the converse we know that D is an L-domain and that $K(D)$ has property M. Given any finite set A of compact elements and any element x of D there is a supremum of the set $\downarrow x \cap A$ in the principal ideal generated by x. Mapping each element onto this supremum is a continuous function, since A consists of compact elements only and suprema of compact elements are again compact in a lattice. The image of this function is finite by property M. This shows that D is isomorphic to a bilimit of finite posets. (A more detailed account of this well known fact can be found in any of the following sources [Plo76, Gun85, Jun88a]. ▮

Since the bifinite L-domains lie at the intersection of two "nice" categories, they inherit some of that niceness themselves:

Proposition 7 *The category of bifinite L-domains and continuous functions is a cartesian closed category.* ▮

Proposition 8 *The category of bifinite L-domains and embedding-projection pairs has bilimits for directed collections.* ▮

4 Building universal domains.

The concept of a "universal domain" dates back at least to Scott's paper [Sco76] on $\mathbf{P}\omega$ and is widely used in the current literature. The term "universal domain" is somewhat vaguely defined, however. We see basically two uses as being the most common. The easiest of these to understand is what one might call a "poor man's universal domain". Typically it is a domain which satisfies an isomorphism

$$V \cong (V \to V) + F_1(V) + \cdots + F_n(V) \tag{1}$$

where $F_1, \ldots, F_n$ are operators over which domain equations must be solved. One often sees such universal domains being used in the type theory literature [MPS84, Car84]. The theory of domains provides us with all of the mathematical tools generally needed for solving equations like (1) so that we may employ such definitions quite freely and confidently. On the other hand, the poor man's universal domain depends on the choice of the functors F_i and it would be nice to know more facts about the order structure of the solution than the existence result for the solution tells us. It is therefore appealing to have a *single* universal domain $\mathcal{U}$ which has *all* domains of interest as retracts. Of course, this is subject to one's interpretation of "domains of interest", but it is not dependent on a commitment to some finite list of functors. We refer the reader to Taylor [Tay87] for a full discussion of universal domains (which he calls "saturated domains"). For the purpose of clarity, let us propose a definition of "universal domain" which will give the reader some idea what we are after.

Definition: Let $\mathbf{C}$ be a category. An object $\mathcal{U}$ is *universal* in $\mathbf{C}$ if it is weakly terminal, *i.e.* for every object A of $\mathbf{C}$, there is a (not necessarily unique) arrow $f: A \to \mathcal{U}$. ▮

The term "universal domain" probably comes from the model theoretic notion of a "universal model" which has a similar definition [CK73]. Universal models can be built using the concept of *saturation* first presented in [Vau61] and it will be our goal below to convert this model-theoretic technique to domain-theoretic ends. Of course, any category that has a terminal object has a universal domain. However, one typically has it in mind that the arrows of the category $\mathbf{C}$ are monics. In particular, we show that the category $\omega\mathbf{BL}^{ep}$ of ω-bifinite L-domains with embedding-projection pairs has a universal domain.

The proof uses techniques from Gunter [Gun87]. However, naively mimicing the construction which appears there will not work. We therefore begin by devising a general theory which can be applied to obtain a universal domain for both $\omega\mathbf{B}^{ep}$ (as described in [Gun87]) and $\omega\mathbf{BL}^{ep}$. We also derive universal domains for $\omega\mathbf{S}^{ep}$ (the category of Scott domains) and $\omega\mathbf{Lat}^{ep}$ (the category of algebraic lattices), which differ from the ones given by Scott in [Sco76, Sco81].

In particular, we provide a categorical treatment of the essential ingredients that make the universal domain construction work. The construction is reminiscent of one from general model theory. For example, fix a first order theory T in a countable language and suppose that T has a countable homogeneous model A. One can show that A is elementarily embedded in a countable model of T as follows. It is easy to see that A is elementarily embedded in a countable model A_1 which is homogeneous with respect to finite sequences taken from A. One can use a similar construction to build a sequence of models A_i such that, for each $j < i$, the model A_i is homogeneous with respect to finite sequences of elements from A_j and A_j is elementarily embedded in A_i. The colimit of this chain will be the desired homogeneous extension of A. The reader can find many constructions that use this basic idea in a standard book on model theory such as [CK73].

We begin with the following concept:

Definition: An arrow $f: A \to B$ is an *increment* if, whenever $f = h \circ g$, then either h or g is an isomorphism. ▮

Perhaps the simplest example of an increment is the inclusion map $f: S \to T$ between finite sets S and T, such that $S = T \cup \{x\}$ for some x. If $\mathbf{C}$ is a poset (considered as a category), then an arrow $x \sqsubseteq y$ is an increment if and only if there is no element of $\mathbf{C}$ between x and y. If we consider the category of L-domains with embedding-projection pairs, then an arrow $s: A \to A'$ from a finite L-domain A into an L-domain A' is an increment if and only if A' has at most one more point than A. Figure 3 indicates a typical increment in this category. The increment embeds a four element poset into a poset with five elements; the closed circle indicates the "new" element.

Figure 3: A typical increment in $\omega\mathbf{BL}^{ep}$. The poset on the left is embedded in the poset on the right. The open circles show the image of the embedding.

An ω-*chain* in a category $\mathbf{C}$ is a functor $F\colon \omega \to \mathbf{C}$ from the ordinal ω (considered as a category) into $\mathbf{C}$. In essence, an ω-chain is a sequence of objects A_i where $i < \omega$ and a collection of arrows $a_{ji}\colon A_i \to A_j$ where $i \leq j < \omega$. For each i, the arrow a_{ii} is the identity on A_i and, for any $i \leq j \leq k$, one has $a_{kj} \circ a_{ji} = a_{ki}$.

Definition: A concrete category $\mathbf{C}$ is *incremental* if

1. $\mathbf{C}$ has an initial object,
2. $\mathbf{C}$ has colimits of ω-chains,
3. every object A of $\mathbf{C}$ is a colimit of an ω-chain (A_i, a_{ij}) where A_0 is initial, each A_i is finite (in the category $\mathbf{C}$) and each arrow $a_{i+1,i}\colon A_i \to A_{i+1}$ is an increment. ∎

For example, the category of countable sets and injections is incremental. When we are taking about incremental categories of domains with embedding-projection pairs we may refer to bilimits rather than colimits. We are especially interested in the following example:

Theorem 9 *The category $\omega\mathbf{B}^{ep}$ of ω-bifinite domains and embedding-projection pairs is incremental.*

Proof: This is Theorem 22 (the Enumeration Theorem) of [Gun87]. ∎

Corollary 10 *The categories $\omega\mathbf{BL}^{ep}$, $\omega\mathbf{S}^{ep}$, and $\omega\mathbf{Lat}^{ep}$ are incremental.*

Proof: Let D be a ω-bifinite L-domain (Scott domain, Lattice) and let (D_i, d_{ij}) be an ω-chain of increments with bilimit D in $\omega\mathbf{B}^{ep}$. By definition, each D_i is embedded in D and must therefore itself be a ω-bifinite L-domain (Scott domain, Lattice). ∎

Let $\mathbf{C}$ be an incremental category and let A be an object of $\mathbf{C}$. An object A^+ and arrow $s: A \to A^+$ is a *relative saturation* of A (or just a *saturation* for short) if, for every increment $f: B \to B'$ and arrow $g: B \to A$, there is an arrow h which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "g"'] & B' \arrow[d, "h"] \\ A \arrow[r, "s"] & A^+ \end{tikzcd}$$

Let us say that an incremental category $\mathbf{C}$ *has finite saturations* if, for every finite object A of $\mathbf{C}$, there is a saturation $s: A \to A^+$ where A^+ is finite.

Theorem 11 *If an incremental category has finite saturations, then it has a universal object.*

Proof: Suppose $\mathbf{C}$ is an incremental category with finite saturations. Let S_0 be any initial object of $\mathbf{C}$. Build the chain $S_0, S_1 = S_0^+, ..., S_{i+1} = S_i^+, ...$ where $s_{i+1,i}$ is a saturation for each i. Let $\mathcal{U}$ be a bilimit for this chain. We claim that $\mathcal{U}$ is universal. To see this, suppose A is any object of $\mathbf{C}$ and we will demonstrate an arrow $f: A \to \mathcal{U}$. Since $\mathbf{C}$ is incremental, A is the bilimit of a chain (A_i, a_{ij}) of finite objects where A_0 is initial and each arrow $a_{i+1,i}: A_i \to A_{i+1}$ is an increment. Now, there is an arrow $f_0: A_0 \to S_0$ since A_0 is initial. Suppose an arrow $f_i: A_i \to S_i$ is given. Since $a_{i+1,i}$ is an increment and $s_{i+1,i}$ is a saturation, there is an arrow f_{i+1} such that the following diagram commutes:

$$\begin{tikzcd} A_i \arrow[r, "a_{i+1,i}"] \arrow[d, "f_i"'] & A_{i+1} \arrow[d, "f_{i+1}"] \\ S_i \arrow[r, "s_{i+1,i}"] & S_{i+1} \end{tikzcd}$$

This collection of arrows f_i gives rise to a cocone with vertex $\mathcal{U}$ over the chain (A_i, a_{ij}) whose vertex is $\mathcal{U}$. Since A is a bilimit of this chain, there must consequently be a mediating arrow $f: A \to \mathcal{U}$ as desired. ▮

Thus, to prove that there is a universal object in the category of ω-bifinite domains (as was done in [Gun87]) or that of ω-bifinite L-domains, it suffices to demonstrate that the category in question has finite saturations. The fact that $\omega\mathbf{B}^{ep}$ has finite saturations is proved in [Gun87]. We show how to derive this result for $\omega\mathbf{B}^{ep}$, $\omega\mathbf{BL}^{ep}$, $\omega\mathbf{S}^{ep}$, and $\omega\mathbf{Lat}^{ep}$ in the next section. By Theorem 11 this will prove:

Theorem 12 *The following categories have universal domains:*

1. $\omega\mathbf{B}^{ep}$

2. $\omega\mathbf{BL}^{ep}$

3. $\omega\mathbf{S}^{ep}$

4. $\omega\mathbf{Lat}^{ep}$ ∎

5 Constructing saturations.

In this section we will construct finite saturations for $\omega\mathbf{B}^{ep}$, $\omega\mathbf{BL}^{ep}$, $\omega\mathbf{S}^{ep}$, and $\omega\mathbf{Lat}^{ep}$. We saw earlier that if D is a finite poset then an increment $f\colon D \to D'$ adds at most one point x_f to D. The idea for constructing a saturation D^+ is to take all points which may be added by an increment.

Since each increment $f\colon D \to D'$ corresponds to a unique projection $g\colon D' \to D$, there is some element $u_f \in D$ onto which x_f is mapped by g. In fact, $f(u_f)$ is the largest element of $f(D)$ below x_f. Similarly, the set $\uparrow x_f \cap f(D)$ corresponds to an upper set U_f in D. This suggests the following definition for a finite poset $D \in \omega\mathbf{B}^{ep}$:

$$D^+ = \{(u, U) \mid u \in D, u \sqsubseteq U = \uparrow U \subseteq D\},$$

with the intended meaning that (u, U) stands for a new element x_f just above $u = u_f$ and below all elements of $U = U_f$. Obviously there cannot be any new element between u and $\uparrow u$, so the pairs $(d, \uparrow d), d \in D$ represent D inside D^+.

We have to be a little bit more careful in defining D^+ for L-domains, however. Recall that D is an L-domain if and only if each bounded nonempty subset of D has a global meet. A new element added by an increment must not destroy this property. This implies that if x_f is a new element added to D by an increment in $\omega\mathbf{BL}^{ep}$ and if $d, d' \sqsubseteq b$ are contained in U_f then x_f is a lower bound for $\{d, d'\}$ and must consequently be below or directly above $d \sqcap d'$. This says that $d \sqcap d'$ must belong to U_f or it must be equal to u_f. We add this property to the definition of D^+ for finite L-domains D:

$$\begin{aligned} D \in \omega\mathbf{BL}^{ep} : D^+ \quad = \quad & \{(u, U) \mid D \ni u \sqsubseteq U = \uparrow U \subseteq D \\ & \text{and } \{u\} \cup U \text{ is closed under bounded} \\ & \text{nonempty meets.}\}. \end{aligned}$$

Similarly for the two remaining categories:

$$\begin{aligned} D \in \omega\mathbf{S}^{ep} : D^+ \quad &= \quad \{(u, U) \mid D \ni u \sqsubseteq U = \uparrow U \subseteq D \\ & \qquad \text{and } \{u\} \cup U \text{ is closed under nonempty meets.}\}, \\ D \in \omega\mathbf{Lat}^{ep} : D^+ \quad &= \quad \{(u, U) \mid D \ni u \sqsubseteq U = \uparrow U \subseteq D \\ & \qquad \text{and } \{u\} \cup U \text{ is closed under meets.}\}. \end{aligned}$$

The order on D^+ is defined uniformly by

$$(u, U) \leq (v, V) \Leftrightarrow v \in U \text{ or } (v = u \text{ and } V \subseteq U).$$

Note that $(u, U) \leq (v, V)$ implies $u \sqsubseteq v$ and $V \subseteq U$, so $\leq$ is indeed a partial order on D^+. It is also helpful to recognize that for a given $u \in D$ the set of all $U \subseteq D$ such that $(u, U) \in D^+$, is a lattice. This follows from the observation that $(u, \uparrow u) \in D^+$ and that if $(u, U_1), (u, U_2) \in D^+$ then $(u, U_1 \cap U_2) \in D^+$. We denote the smallest set U which contains a set $X \subseteq D$ and for which (u, U) belongs to D^+ by $\langle X \rangle_u$.

Lemma 13 *If D is a finite L-domain (bounded-complete domain, lattice) then so is D^+.*

Proof: We have to show that D^+ has infima for bounded sets. So let $(u, U), (u', U') \leq (v, V)$ be three elements in D^+. Since $\{u, u'\}$ is bounded by v, the infimum $u \sqcap u'$ exists in D. The corresponding upper set U'' must at least contain U and U' but depending on whether $u \sqcap u'$ is contained in $\{u, u'\}$ or not it may be necessary to include also u and/or u'. We can express this as follows: $U'' = \langle U \cup U' \cup (\{u, u'\} \setminus \{u \sqcap u'\}) \rangle_{u \sqcap u'}$. If (w, W) is any other lower bound then either $w = u \sqcap u'$ or $w < u \sqcap u'$. In the first case W must contain U'' as we took U'' as small as possible. In the second case, W must contain u and u' and hence also $u \sqcap u'$.

The proof for Scott-domains is the same with the single difference that $\{(u, U), (u', U')\}$ is not necessarily bounded. In order to show that D^+ is a lattice if D belongs to $\omega\mathbf{Lat}^{ep}$ it suffices to note that $(\top, \phi)$ is the largest element of D^+. ▮

Lemma 14 *If D is a finite poset (L-domain, Scott-domain, lattice) then D^+ is a saturation for D in the respective category.*

Proof: We indicated above that D is embedded in D^+ via the mapping $d \mapsto (d, \uparrow d)$. The corresponding projection is given by $(u, U) \mapsto u$.

Let $f\colon D \to D'$ be an increment and let $u_f \in D$ and $U_f \subseteq D$ be defined as above. In the definition of D^+ we already argued that (u_f, U_f) belongs to D^+ in all four cases. It therefore remains to show that D' is embedded in D^+. We identify D' with the subset $\{(d, \uparrow d) \mid d \in D\} \cup \{(u_f, U_f)\}$ of D^+. For each $(u, U) \in D^+$ there is a largest element of D' below it: if $(u_f, U_f) \leq (u, U)$ then either $u = u_f$, in which case (u_f, U_f) is the largest element of $\downarrow(u, U) \cap D'$, or u is contained in U_f. In the latter case we have that $(u_f, U_f) \leq (u, \uparrow u) \leq (u, U)$ and $(u, \uparrow u)$ is the largest element of D' below (u, U). Hence there is a projection from D^+ onto D'. ▮

An illustration of the four different constructions can be found in Figure 4 at the end of the paper. The reader is challenged to check that the figure labelled A^+ in $\omega\mathbf{B}^{ep}$ is, in fact, not an L-domain whereas the figure labelled A^+ in $\omega\mathbf{BL}^{ep}$ is one. Similarly, the figure labelled B^+ in $\omega\mathbf{BL}^{ep}$ is not a Scott domain although the figure to its right is a Scott domain. The third trio of examples is a similar illustration for algebraic lattices.

6 Saturated domains.

We hope that the reader can now appreciate how Theorem 11 can be used to demonstrate the existence of a universal object. In the proof of that theorem, there is a construction of a universal domain using the saturations that exist in the category. Since a given finite object may have many non-isomorphic saturations, it is possible that the construction used there may give different universal domains if one uses different saturations. In this section we demonstrate that this is not the case in a category of ω-bifinite domains: *regardless of the choice of saturations, the construction in Theorem 11 is unique up to isomorphisms.* In particular, we will define the notion of a *saturated* domain by analogy with the concept of a saturated model of a first order theory [CK73]. We then show, as one shows the corresponding model-theoretic result, that there is a unique saturated domain up to isomorphism. It is then shown that the universal domain constructed in Theorem 11 is, in fact, saturated. This shows that there is a "canonical" choice of universal domain for many of the categories of domains used in denotational semantics [GS88]. It is remarked that the bounded complete universal domain of Scott [Sco81, Sco82a, Sco82b] is *not* saturated and is therefore not isomorphic to the universal bounded complete domain constructed in the previous section.

As an abbreviation, let us refer to an incremental full sub-category $\mathbf{C} \subseteq \omega\mathbf{B}^{ep}$ as a *category of domains* if it is closed under embeddings: *i.e.* if $E \in \mathbf{C}$, $D \in \omega\mathbf{B}$ and there is an embedding-projection pair $f\colon D \to E$, then D is in $\mathbf{C}$. The key concept of this section is given in the following:

Definition: Let $\mathbf{C}$ be a category of domains. An object $U \in \mathbf{C}$ is *fully saturated in* $\mathbf{C}$ (or *saturated*, for short) if, for every pair of finite domains M, N and embedding-projection pairs, $f\colon M \to U$ and $g\colon M \to N$, there is a (not necessarily unique) embedding-projection pair h which completes the following diagram:

$$\begin{tikzcd} M \arrow[d, "g"'] \arrow[dr, "f"] & \\ N \arrow[r, dashed, "h"] & U \end{tikzcd}$$

Theorem 15 *Let U be a fully saturated object in a category $\mathbf{C}$ of domains. Then U is universal for $\mathbf{C}$.*

Proof: Each ω-bifinite domain D is the bilimit of an ω-chain (A_i, a_{ij}) of finite posets. We may assume that $A_0 = \{\perp\}$. Clearly, A_0 is embedded in $U \in \mathbf{C}$ and so by definition $A_1, A_2, \ldots$ are embedded in U. This cocone over U gives rise to an embedding $g\colon D \to U$. ▮

To prove the desired results about saturated domains, it is useful to introduce a few notations and facts which are useful in dealing with categories of domains. If $f: D \to E$ is an embedding-projection pair and f^L is an inclusion map then we write $D \lhd E$. The following lemma is easy to prove and will be used implicitly in the proof of the theorem below:

Lemma 16 *1. If M is a finite poset such that $M \lhd U$, then $M \subseteq K(U)$.*

2. If D is ω-bifinite and $S \subseteq K(D)$ is finite, then there is a finite $N \lhd D$ such that $S \subseteq N$.

3. If $M \lhd D$, $N \lhd D$ and $M \subseteq N$, then $M \lhd N$. ▮

Lemma 17 *Let U be an object in a category of domains. If U is saturated, then for every finite $M \lhd U$ and embedding-projection pair $f: M \to N$ into a finite poset N, there is a poset $N' \lhd U$ such that $N \cong N'$.* ▮

Proof: Let N' be the image under the embedding h whose existence is guaranteed by definition. ▮

Theorem 18 *If a category of domains has a saturated object, then it is unique up to isomorphism.*

Proof: Let $\mathbf{C}$ be a category of domains and suppose that U and V are saturated objects of $\mathbf{C}$. Let $u_0, u_1, \ldots$ and $v_0, v_1, \ldots$ be enumerations of the bases of U and V respectively. Assume that $u_0 = \bot_U$ and $v_0 = \bot_V$. We construct an isomorphism between $K(U)$ and $K(V)$ by a "back and forth" construction. The first partial isormorphism is the unique arrow $f_0: \{u_0\} \cong \{v_0\}$. Suppose now that we have finite posets $L \lhd U$ and $L' \lhd V$ such that there is an isomorphism $f_{n-1}: L \cong L'$. Suppose further that $\{u_0, \ldots, u_{n-1}\} \subseteq L$ and $\{v_0, \ldots, v_{n-1}\} \subseteq L'$. We wish to extend the isormorphism f_{n-1} to an isomorphism $f_n: M \cong M'$ where $M \lhd U$ and $M' \lhd V$ are finite and $u_n \in M$ and $v_n \in M'$. Now, we know that there is a finite poset $N \lhd U$ with $L \cup \{u_n\} \subseteq N$. From the inverse of the isomorphism f_{n-1} we can build an embedding-projection pair $f: L' \to N$. Since V is saturated, there is a poset $N' \lhd V$ and an isomorphism $g: N' \cong N$. To complete the argument, we add $\{v_n\}$ to N' and find a subset $M' \subseteq V$ such that $\{v_n\} \cup N' \subseteq M'$. Since U is saturated we find an isomorphic copy M of M' inside U, containing L, such that the isomorphism $g^{-1}: N \cong N'$ is extended to an isomorphism $f_n: M \cong M'$. In this way we obtain a sequence $f_0, f_1, \ldots$ of isomorphisms whose union is an isomorphism between $K(U)$ and $K(V)$. This isomorphism extends to an isomorphism between U and V. ▮

Theorem 19 *If an incremental category of domains has finite saturations, then it has a saturated object.*

Proof: Recall the construction in the proof of Theorem 11. Suppose $\mathbf{C}$ is an incremental category with finite saturations. Let S_0 be any initial object of $\mathbf{C}$. Build the chain $S_0, S_1 = S_0^+, \ldots, S_{i+1} = S_i^+, \ldots$ where $s_{i+1,i}$ is a saturation for each i. Let $\mathcal{U}$ be a bilimit for this chain. It will simplify matters to assume that each of these saturations is an inclusion by replacing each S_i by its embedded image in $\mathcal{U}$. Suppose $M \lhd U$ and there is an embedding-projection pair $f: M \to N$ for some finite $N \in \mathbf{C}$. We must show that there is is an h such that

$$\begin{tikzcd} M \arrow[d, "f"'] \arrow[dr, "\lhd"] & \\ N \arrow[r, dashed, "h"] & U \end{tikzcd}$$

The proof is by induction on the number n of elements of N not in the image of f. If $n = 0$, then f is an isomorphism so we may take the coextension of f^{-1} to U as h. If $n \geq 1$, then it is possible to find an increment $f': M \to N'$ such that f' extends f and $N' \lhd N$ and there is exactly one element in N' which is not in the image of f'. Since M is finite, there is an i such that $M \subseteq S_i$. Since f' is an increment, there is an h' such that

$$\begin{tikzcd} M \arrow[d, "f'"'] \arrow[dr, "\lhd"] & \\ N' \arrow[r, dashed, "h'"] & S_{i+1} = S_i^+ \end{tikzcd}$$

We can now apply our inductive hypothesis to find an h such that

$$\begin{tikzcd} N' \arrow[d, "\lhd"'] \arrow[dr, "h'"] & \\ N \arrow[r, dashed, "h"] & U \end{tikzcd}$$

By putting these last two diagrams together we see that h has the desired properties. ▮

Corollary 20 *There are saturated objects in each of the following categories:*

1. $\omega\mathbf{B}^{ep}$
2. $\omega\mathbf{BL}^{ep}$
3. $\omega\mathbf{S}^{ep}$

4. $\omega\mathbf{Lat}^{ep}$ ∎

It is interesting to note that Scott's universal domain for the consistently complete domains [Sco81, Sco82a, Sco82b] is *not* saturated. To see this, it suffices to note that the meet of compact elements in the saturated consistently complete domain is not compact whereas the intersection of compact elements in Scott's universal domain *is* compact.

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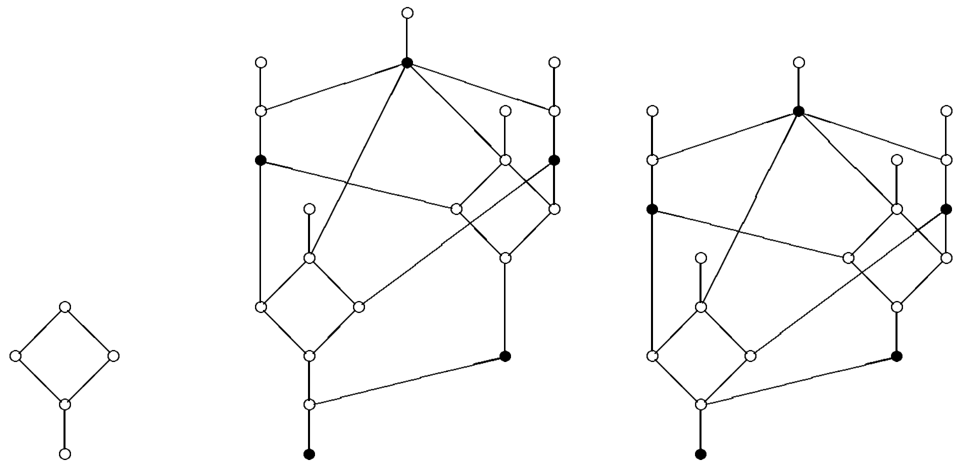
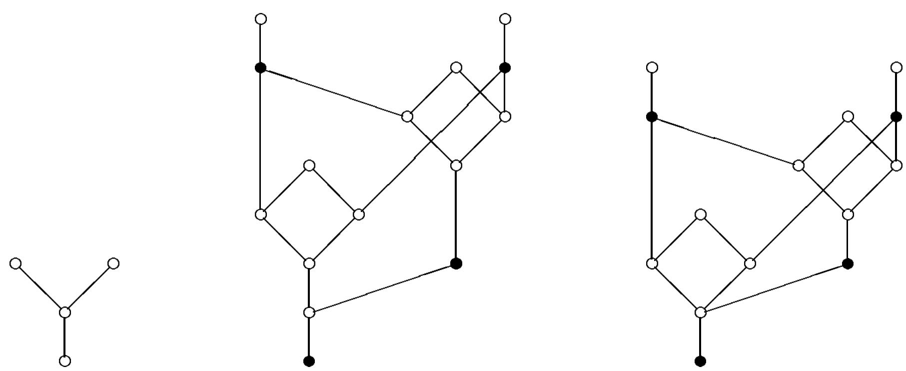
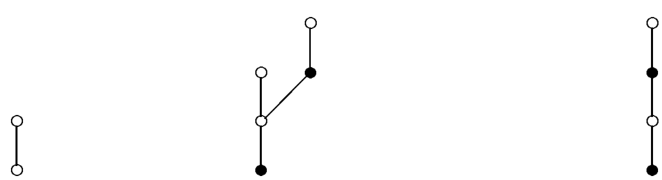


A $\quad$ A^+ in $\mathbf{SAdom}^{ep}$ $\quad$ A^+ in $\mathbf{SALdom}^{ep}$



B $\quad$ B^+ in $\mathbf{SALdom}^{ep}$ $\quad$ B^+ in $\mathbf{Sdom}^{ep}$



C $\quad$ C^+ in $\mathbf{Sdom}^{ep}$ $\quad$ C^+ in $\mathbf{Lat}^{ep}$

Figure 4: Saturations in different categories.